# Phys 374 Final Exam : Solution

1. a.) $\vec{\nabla}\left(\frac{6}{r^2+3}\right) = \frac{-6}{(r^2+3)^2}\vec{\nabla}r^2 = \frac{-12\vec{r}}{(r^2+3)^2}$

b.) $\int_A^H \vec{\nabla}f\cdot d\vec{r} = f(\vec{r}_H) - f(\vec{r}_A)$ $\qquad \vec{r}_H^{\,2} = 3^2+3^2+3^2 = 27 \qquad \vec{r}_A^{\,2} = 1^2+1^2+0^2 = 2$

$$= \frac{6}{27+3} - \frac{6}{2+3} = \frac{1}{5} - \frac{6}{5} = \underline{\underline{-1}}$$

2.) $$\int_B^E \vec{v}\cdot d\vec{r} = \int_2^3 v_x\,dx\Big|_{\substack{y=1\\z=0}} + \int_0^{-1} v_z\,dz\Big|_{\substack{x=3\\y=1}} + \int_1^2 v_y\,dy\Big|_{\substack{x=3\\z=-1}}$$

$$= \int_2^3 -a\,dx + \int_0^{-1} c(3+1)\,dz + \int_1^2 b\,3\,dy = \underline{-a-4c+3b}$$

3.) $$E^2 = \frac{m^2c^4}{1-v^2/c^2} \qquad \text{let } x = v^2/c^2 \ll 1$$

$$= \frac{m^2c^4}{1-x} = m^2c^4\left(1 + x + x^2 + x^3 + \dots\right)$$

$$= m^2c^4 + m^2c^2v^2 + m^2v^4 + \frac{m^2v^6}{c^2} + \dots$$

4.) $$\oint_{ABCA} \vec{v}\cdot d\vec{r} = \int_S da\,\underbrace{\hat{n}}_{-\hat{e}_z}\cdot(\vec{\nabla}\times\vec{v}) = \int_0^2 dx\int_x^2 dy\,(-xy) = -\int_0^2 dx\,x\int_x^2 dy\,y$$

$$= -\int_0^2 dx\,x\left(\tfrac{1}{2}y^2\right)\Big|_x^2 = -\int_0^2 dx\,x\left(2-\tfrac{1}{2}x^2\right) = -x^2 + \frac{x^4}{8}\Big|_0^2 = -4+2$$

$$= \underline{\underline{2}}$$

5.) a1.) $\displaystyle\int \vec{V}\cdot\hat{n}\,ds = \int_r c\,dV = \underline{\underline{c\cdot\frac{4\pi}{3}r^3}}$ $\quad (r<R)$

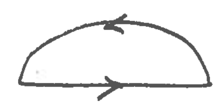
a2.) $\displaystyle\int \vec{V}\cdot\hat{n}\,ds = c\int_R dV = \underline{\underline{c\cdot\frac{4\pi}{3}R^3}}$ $\quad (r>R)$

b.) $\displaystyle\int_{cube} \vec{V}\cdot\hat{n}\,ds = c\int_R dV = \underline{\underline{c\cdot\frac{4\pi}{3}R^3}}$ $\quad$ (sphere is inside cube)

d.) if $\vec{V}=\vec{E}$ & $\vec{\nabla}\cdot\vec{E}=\dfrac{\rho}{\epsilon_0}$ where $\rho = \dfrac{Q}{\frac{4\pi}{3}R^3} = \dfrac{\text{charge}}{\text{volume}}$

then $\underline{\underline{c = \dfrac{3Q}{4\pi\epsilon_0 R^3}}}$

6.) $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{ikz}\frac{1}{(z+ia)(z-ia)}$

for $k>0$, use $C_+$ 

for $k<0$, use $C_-$ 

$\underline{k>0}$ $\quad \displaystyle\frac{1}{2\pi}\cdot 2\pi i\left.\frac{e^{ikz}}{z+ia}\right|_{z=ia} = \frac{1}{2a}e^{-ka}$

$\underline{k<0}$ $\quad \displaystyle\frac{1}{2\pi}\cdot(-2\pi i)\left.\frac{e^{ikz}}{z-ia}\right|_{z=-ia} = \frac{1}{2a}e^{ka}$

$$\left.\begin{matrix} k>0 \\ k<0 \end{matrix}\right\} = \underline{\underline{\frac{1}{2a}e^{-|k|a}}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

7.) Even function $\Rightarrow$ all $b_n = 0$

$$\int_{-a}^{a} dx \to 2\int_0^a dx$$

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} dx\, f(x) = \frac{1}{L}\int_0^a dx\left(1 - \frac{x}{a}\right) = \frac{1}{L}\left(x - \frac{x^2}{2a}\right)\Bigg|_0^a = \frac{a}{2L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} dx\, f(x)\cos\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\int_0^a dx\left(1 - \frac{x}{a}\right)\cos\left(\frac{n\pi x}{L}\right)$$

$u = x \quad dv = dx\cos\left(\frac{n\pi x}{L}\right) \quad v = \frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}$

$$= \frac{2}{L}\,\frac{1}{\frac{n\pi}{L}}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^a - \frac{2}{aL}\,x\,\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\Bigg|_0^a$$

$$+ \frac{2}{aL}\,\frac{1}{\frac{n\pi}{L}}\int_0^a dx\,\sin\left(\frac{n\pi x}{L}\right)$$

$$= \underbrace{\frac{2}{n\pi}\sin\left(\frac{n\pi a}{L}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi a}{L}\right)}_{=0} - \frac{2}{an\pi}\,\frac{1}{\frac{n\pi}{L}}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^a$$

$$= -\frac{2L}{a}\,\frac{1}{(n\pi)^2}\left(\cos\left(\frac{n\pi a}{L}\right) - 1\right)$$

$$a_n = \frac{2L}{a}\,\frac{1}{(n\pi)^2}\left(1 - \cos\left(\frac{n\pi a}{L}\right)\right)$$

so $$f(x) = \frac{a}{2L} + \frac{2L}{a}\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}\left(1 - \cos\left(\frac{n\pi a}{L}\right)\right)\cos\left(\frac{n\pi x}{L}\right)$$

8.) a.) $$V(t) - R\frac{dQ}{dt} - L\frac{d^2Q}{dt^2} - \frac{Q}{C} = 0$$ $\quad O(t) = -L\frac{d^2Q}{dt^2}$

b.) $$V(\omega) + \left(i\omega R + L\omega^2 - \frac{1}{C}\right)Q(\omega) = 0$$

$$O(\omega) = \omega^2 L\, Q(\omega)$$

so $$O(\omega) = \frac{L\omega^2}{L\omega^2 + i\omega R - 1/C}\,V(\omega)$$

UNIVERSITY OF MARYLAND

Department of Physics
Physics 374 Spring 2009

**Final Examination, Monday, May 18, 2009**

You may provide a **4-digit number** of your choice, but not starting with 0, and request to have your **grade posted** on the web site. Place your number and request under your signature on the exam book

1.) For the function

$$f(r) = \frac{6}{r^2+3}, \qquad (1)$$

where $r = \sqrt{x^2+y^2+z^2}$ is the radius in three dimensions,

a.) (20 pts.) evaluate $\nabla f(r)$.

b.) (20 pts.) Evaluate the line integral $\int \nabla f(r) \cdot d\mathbf{r}$, for $f(r)$ defined in Eq. (1), using the path $ABCDEFGH$ shown in Figure 1. Coordinates $(x, y, z)$ of the points are shown in Figure 1.

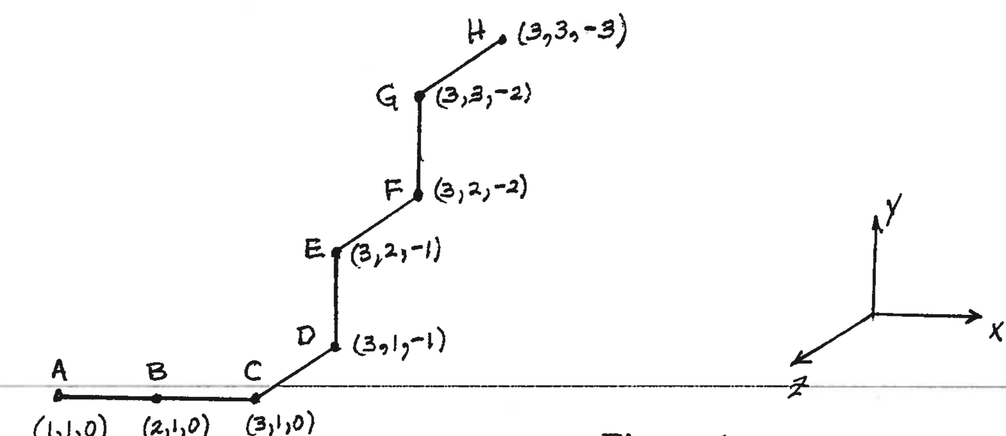



**Figure 1**

2.) (20 pts.) For a vector field $\mathbf{v} = -ay\hat{e}_x + bx\hat{e}_y + c(x+y)\hat{e}_z$, evaluate the line integral $\int \mathbf{v} \cdot d\mathbf{r}$ for the *segment* $BCDE$ of the path shown in Figure 1. *Note:* $\mathbf{v}$ *is not the gradient of any scalar field.*

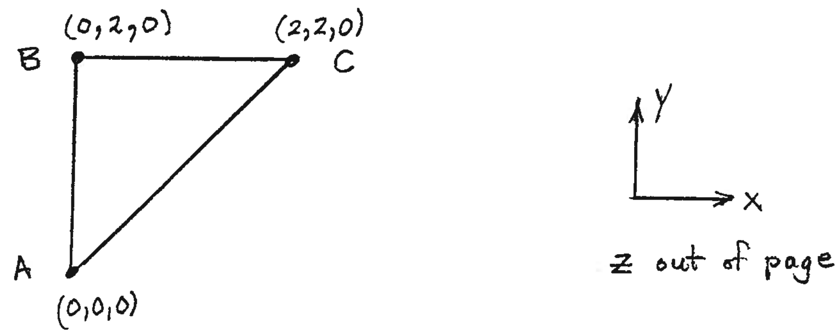
**3.)** (20 pts.) The function

$$E^2 = \frac{m^2c^4}{1 - v^2/c^2} \tag{2}$$

is the square of the relativistic total energy of a particle of mass $m$ and velocity $\mathbf{v}$. Assuming that $v < c$, where $c$ is the speed of light, expand $E^2$ in powers of v including terms up to order $v^6$.

**4.)** (20 pts.) The vector field $\mathbf{v}(\mathbf{r})$ obeys $\nabla \times \mathbf{v} = xy\hat{\mathbf{e}}_z$.

**a.)** Evaluate $\int_C \mathbf{v} \cdot \mathbf{r}$ for the path $ABCA$ around the triangle shown in Figure 2. Coordinates $(x, y, z)$ are shown for each corner of the triangle and the hypotenuse is a segment of the line $x = y$.




**5.)** In the following, $\nabla \cdot \mathbf{v} = c$ is constant inside a sphere of radius $R$ with its center at the origin. Outside the sphere $\nabla \cdot \mathbf{v} = 0$.

**a.)** (10 pts.) Determine the surface integral $\int \mathbf{v} \cdot \hat{\mathbf{n}} dS$ for the surface of a sphere with radius $r < R$ and center at the origin.

**a.)** (10 pts.) Determine the surface integral $\int \mathbf{v} \cdot \hat{\mathbf{n}} dS$ for the surface of a sphere with radius $r > R$ and center at the origin.

**b.)** (10 pts.) Determine the surface integral $\int \mathbf{v} \cdot \hat{\mathbf{n}} dS$ for the surface of a cube with sides of length $2R$ and center at the origin.

**d.)** (10 pts.) If the vector field is the electric field obeying $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, and the total charge $Q$ is distributed with a constant charge density inside the sphere of radius $R$, determine the constant $c$ in terms of $R$, $\epsilon_0$ and other constants.

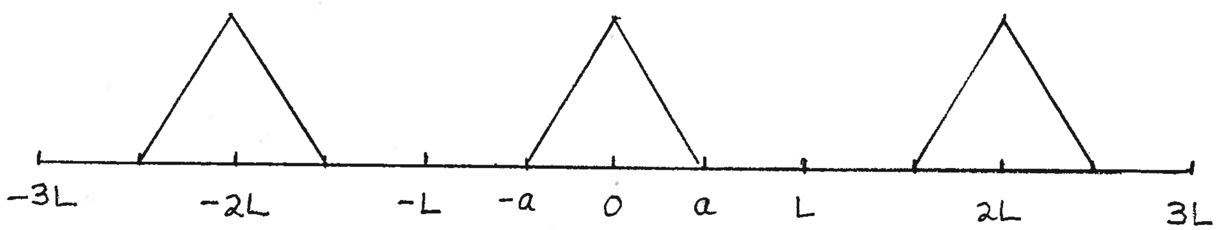
**6.)** (20 pts.) Evaluate the following Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ikx}\frac{1}{x^2+a^2}, \tag{3}$$

**7.)** (20 pts.) Find the Fourier series for a periodic function obeying $f(x+2L)=f(x)$, that is defined in the interval $-L<x<L$ as follows,

$$\begin{aligned} f(t) &= 0, \quad -L<x<-a; \\ f(t) &= 1-\frac{|x|}{a}, \; -a<x<a; \\ f(t) &= 0, \quad a<x<L. \end{aligned} \tag{4}$$

This function is shown in the sketch below.




**8.)** For a RLC circuit as in the sketch below, where $R$ is the resistance, $L$ is the inductance and $C$ is the capacitance,

**a.)** (10 pts.) determine the equation in time domain for the charge, $Q(t)$, that flows in the circuit because of the applied voltage $v(t)$. The charge is related to the current by $I(t)=\frac{dQ(t)}{dt}$.

**b.)** (10 pts.) Determine the relation between the Fourier transform of the output voltage across the inductor, $o(t)$, and the Fourier transform of the input voltage $v(t)$, using the convention

$$v(t)=\int_{-\infty}^{\infty} dt e^{-i\omega t} V(\omega). \tag{5}$$