HW 10 Due 4/29/09

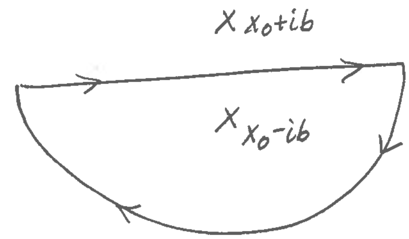
1.) $$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx} \frac{G}{(x-x_0)^2+b^2}$$

$$\rightarrow \frac{1}{2\pi}\int_C dz\; e^{-ikz} \frac{G}{(z-x_0)^2+b^2}$$

$$e^{-ikz} = e^{-ik(x+iy)} = e^{-ikx}\, e^{ky} \longrightarrow 0$$

for $ky \to -\infty$

For $k>0$, use $C_- =$



poles $$\frac{1}{(z-x_0)^2+b^2} = \frac{1}{(z-x_0+ib)(z-x_0-ib)}$$

this pole is at $z = x_0 - ib$

because clockwise

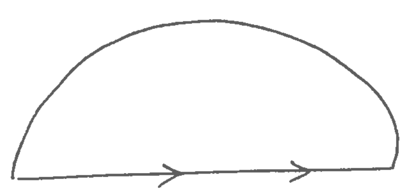
$$I = \frac{1}{2\pi}(-2\pi i)\, \text{Res}\,(z = x_0 - ib)$$

$$= i \left. \frac{\cancel{(z-x_0+ib)}\; e^{-ikz}}{\cancel{(z-x_0+ib)}(z-x_0-ib)} \right|_{z=x_0-ib}$$

$$= -i\,\frac{e^{-ik(x_0-ib)}}{-2ib}$$

$$I = \frac{e^{-ikx_0}\,e^{-kb}}{2b}$$

For $k<0$, use C_+



$$I = \frac{1}{2\pi}(2\pi i)\,\text{Res}\,(z = x_0+ib) = i\left.\frac{e^{-ikz}}{z-x_0+ib}\right|_{z=x_0+ib} = \frac{e^{ikx_0}\,e^{kb}}{2b}$$

For either $k>0$ or $k<0$

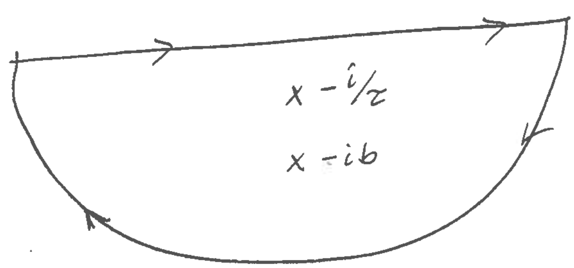
$$\boxed{I = \frac{e^{-i|k|x_0}\, e^{-|k|b}}{2b}}$$

2.) $$I = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \frac{\omega b^2}{\left(\omega + \frac{i}{\tau}\right)(\omega + ib)(\omega - ib)}$$

For $t>0$, close contour with $Im(\omega) < 0$, i.e., C_-

$$I = \int_{C_+} dz\, e^{-izt} \frac{z b^2}{(z + i/\tau)(z+ib)(z-ib)}$$

$$e^{-izt} = e^{-i(x+iy)t} = e^{-ixt} e^{yt} \longrightarrow 0 \quad \text{when } y \to -\infty$$



$$I = -2\pi i \left(Res\left(z = -i/\tau\right) + Res\left(z = -ib\right) \right)$$

$$= -2\pi i \left(\left.\frac{z b^2 e^{-izt}}{z^2+b^2}\right|_{z=-i/\tau} + \left.\frac{z b^2 e^{-izt}}{(z + i/\tau)(z - ib)}\right|_{z=-ib} \right)$$

$$= -2\pi i \left(\frac{-i/\tau\, b^2 e^{-t/\tau}}{b^2 - 1/\tau^2} + \frac{(-ib)\, b^2 e^{-bt}}{(-ib + i/\tau)(-2ib)} \right)$$

$$I = -\frac{2\pi}{\tau} \frac{b^2}{b^2 - 1/\tau^2} e^{-t/\tau} + \frac{\pi b^2 e^{-bt}}{b - 1/\tau}$$

3 a. $\quad O(\omega) = \dfrac{\omega}{\omega + \frac{i}{RC}} V(\omega)$ from 3a.) of HW 9

$$O(t) = \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \frac{\omega}{\left(\omega + \frac{i}{RC}\right)} \frac{\omega_c^2}{(\omega^2 + \omega_c^2)}$$

This is the same integral as in 2.) with

$b = \omega_c$, $\tau = RC$. So the answer is

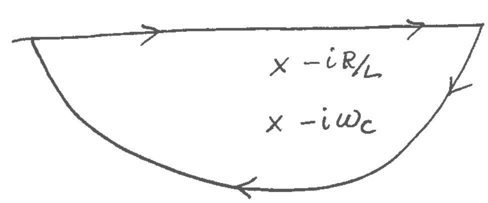
$$O(t) = -\frac{2\pi}{RC} \frac{\omega_c^2}{\omega_c^2 - \frac{1}{R^2C^2}} e^{-t/RC} + \frac{\pi \omega_c^2 e^{-\omega_c t}}{\omega_c - 1/RC}$$

3b.) $\quad O(\omega) = \dfrac{iR/L}{\omega + iR/L} \dfrac{\omega_c^2}{\omega^2 + \omega_c^2}$

$$o(t) = \int_{-\infty}^{\infty} dz \, e^{-izt} \frac{iR/L}{(z + iR/L)} \frac{\omega_c^2}{(z + i\omega_c)(z - i\omega_c)}$$

For $t > 0$, $e^{-izt} = e^{-i(x+iy)t} = e^{-ixt} e^{yt} \to 0$ for $y \to -\infty$

so close contour in lower ½ plane



$$o(t) = -2\pi i \left(\mathrm{Res}\left(z = -\frac{iR}{L}\right) + \mathrm{Res}\left(-i\omega_c\right) \right)$$

3b. – continued

$$O(t) = -2\pi i \left(\left. \frac{i\frac{R}{L}\omega_c^2 e^{-izt}}{z^2+\omega_c^2} \right|_{z=-i\frac{R}{L}} + \left. \frac{i\frac{R}{L}\omega_c^2 e^{-izt}}{\left(z+\frac{iR}{L}\right)(z-i\omega_c)} \right|_{z=-i\omega_c} \right)$$

$$= \frac{2\pi R}{L}\omega_c^2 \left(\frac{e^{-R/L\, t}}{\omega_c^2 - \frac{R^2}{L^2}} + \frac{e^{-\omega_c t}}{\left(-i\omega_c + \frac{iR}{L}\right)(-2i\omega_c)} \right)$$

$$O(t) = \frac{2\pi R}{L} \frac{\omega_c^2}{\omega_c^2 - \frac{R^2}{L^2}} e^{-R/L\, t} - \frac{\pi R}{L} \frac{\omega_c}{\omega_c - \frac{R}{L}} e^{-\omega_c t}$$

as $\omega_c \to \infty$, $O(t) \to \frac{2\pi R}{L} e^{-R/L\, t} = 2\pi \times$ Green's fn.

3c.)

$$O(\omega) = \frac{i\omega\frac{R}{L}}{\omega^2 + i\omega\frac{R}{L} - \frac{1}{LC}} \frac{\omega_c^2}{\omega^2+\omega_c^2}$$

$$O(t) = \int_{-\infty}^{\infty} dz\, e^{-izt} \frac{iz\frac{R}{L}\omega_c^2}{\left[\left(\omega+\frac{iR}{2L}\right)^2 - b^2\right](\omega+i\omega_c)(\omega-i\omega_c)}$$

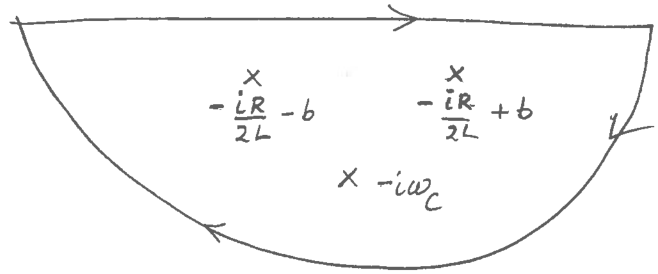
$$b^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2$$

poles at $\omega = -\frac{iR}{2L} \pm b$, $\omega = \pm i\omega_c$

close contour in lower ½ plane $\rightarrow C_-$

$$O(t) = \int_{C_-} dz\, e^{-izt} \frac{iz\frac{R}{L}\omega_c^2}{\left(z+\frac{iR}{2L}-b\right)\left(z+\frac{iR}{2L}+b\right)(z+i\omega_c)(z-i\omega_c)}$$

3c) continued



$$O(t) = (-2\pi i)\left(Res\left(z = -\frac{iR}{2L} - b\right) + Res\left(z = -\frac{iR}{2L} + b\right) + Res\left(z = -i\omega_c\right) \right)$$

$$\left. \frac{e^{-izt}\, iz\frac{R}{L}\omega_c^2}{\left(z + \frac{iR}{2L} - b\right)\left(\omega^2 + \omega_c^2\right)} \right|_{z = -\frac{iR}{2L} - b} = \frac{e^{-\frac{R}{2L}t}\, e^{ibt}\, i\left(-\frac{iR}{2L} - b\right)\frac{R}{L}\omega_c^2}{(-2b)\left[\left(-\frac{iR}{2L} - b\right)^2 + \omega_c^2\right]}$$

$$\left. \frac{e^{-izt}\, iz\frac{R}{L}\omega_c^2}{\left(z + \frac{iR}{2L} + b\right)\left(\omega^2 + \omega_c^2\right)} \right|_{z = -\frac{iR}{2L} + b} = \frac{e^{-R/2L\, t}\, e^{-ibt}\, i\left(\frac{-iR}{2L} + b\right)\frac{R}{L}\omega_c^2}{2b\left[\left(-\frac{iR}{2L} + b\right)^2 + \omega_c^2\right]}$$

$$Res\left(z = -i\omega_c\right) = \left. \frac{e^{-izt}\, iz\frac{R}{L}\omega_c^2}{\left(z^2 + iz\frac{R}{L} - \frac{1}{LC}\right)\left(z - i\omega_c\right)} \right|_{z = -i\omega_c}$$

$$= \frac{e^{-\omega_c t}\, \frac{i}{2}\frac{R}{L}\omega_c^2}{\left(-\omega_c^2 + \omega_c\frac{R}{L} - \frac{1}{LC}\right)}$$

leads to $\left(\text{using } b \equiv \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)$

$$\boxed{\begin{aligned} O(t) = {} & 2\pi\frac{R}{L}\left(\frac{\omega_c^2}{\omega_c^2 + b^2 + 2i\frac{R}{L}b - \frac{R^2}{4L^2}}\right)\frac{\left(b + \frac{iR}{2L}\right)}{2b}\, e^{-R/2L\, t}\, e^{ibt} \\ & 2\pi\frac{R}{L}\left(\frac{\omega_c^2}{\omega_c^2 + b^2 - 2i\frac{R}{L}b - \frac{R^2}{4L^2}}\right)\left(\frac{b - \frac{iR}{2L}}{2b}\right) e^{-R/2L\, t}\, e^{-ibt} - \frac{\pi\frac{R}{L}\omega_c^2\, e^{-\omega_c t}}{\omega_c^2 + \frac{1}{LC} - \omega_c\frac{R}{L}} \end{aligned}}$$