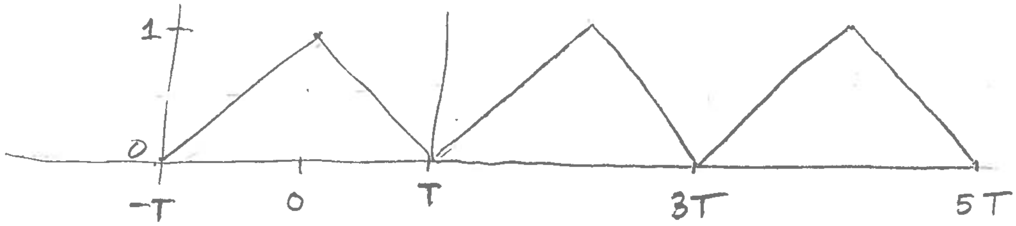
Find the Fourier ~~A~~ time series for $f(t) = 1 - t/T$, $0 < t < T$ **HW 9 Due 4/22/09**

$= 1 + t/T \quad -T < t < 0.$

$\Rightarrow f(t) = 1 - \frac{|t|}{T}$



$f(t)$ is an even function, so $b_n = 0$. The average of $f(t) = 1/2$, so $\frac{a_0}{2} = \frac{1}{2}$

$$f(t) = \frac{a_0}{2} + \sum_n a_n \cos\left(\frac{n\pi t}{T}\right)$$

where $a_n = \frac{1}{T}\int_{-T}^{T} dt \left(1 - \frac{|t|}{T}\right) \cos\left(\frac{n\pi t}{T}\right)$

$$= \frac{2}{T}\int_0^T dt \left(1 - \frac{t}{T}\right)\cos\left(\frac{n\pi t}{T}\right)$$

(because integrand is even, integrate 0 to T & multiply by 2)

$$= \frac{2}{T}\left[\frac{T}{n\pi}\sin\left(\frac{n\pi t}{T}\right)\Big|_0^T\right] - \frac{2}{T^2}\int_0^T u\,dv$$

$u = t$

$dv = \frac{T}{n\pi}\sin\left(\frac{n\pi t}{T}\right)$

$$= \frac{2}{n\pi}\underbrace{\left[\sin(n\pi) - \sin(0)\right]}_{0} - \underbrace{\frac{2}{T^2}\frac{tT}{n\pi}\sin\left(\frac{n\pi t}{T}\right)\Big|_0^T}_{0}$$

$$+ \frac{2}{T^2}\int_0^T dt\, \frac{T}{n\pi}\sin\left(\frac{n\pi t}{T}\right)$$

$$= \frac{2}{T^2}\frac{T}{n\pi}\left[\frac{-T}{n\pi}\cos\left(\frac{n\pi t}{T}\right)\right]\Big|_0^T$$

$$= \frac{-2}{n^2\pi^2}\cos\left(\frac{n\pi t}{T}\right)\Big|_0^T$$

$$= -\frac{2}{n^2\pi^2}\left((-1)^n - 1\right) = \frac{4}{n^2\pi^2} \text{ if } n = \text{odd}, \quad 0 \text{ otherwise}$$

$$\boxed{f(t) = \frac{1}{2} + \sum_{n\ \mathrm{odd}} \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi t}{T}\right)}$$

(Debug) In[121]:=

```
Array[ an, 100];
Array[an1, 100];
Array[ kh, 100];
Array[ kp, 100];
max = 100;
length = 8;
h = length / 1000.;
For[n = 1, n < max, n += 2,
  an[n] = 4. / (n Pi);
  an1[n] = 4. / Power[n Pi, 2];
  kp[n] = n Pi / length;
  kh[n] = kp[n] h;
 ];
(*Print[" an : ",an[1],"  ",an[3],"  ",an[5]]
 Print[" an1 : ",an1[1],"  ",an1[3],"  ",an1[5]]
 Print[" kp : ",kp[1],"  ",kp[3],"  ",kp[5]]
 Print[" kh : ",kh[1],"  ",kh[3],"  ",kh[5]]*)

Plot[Sum[an1[i] * Cos[kp[i] * x] , {i, 1, 3, 2}] + .5, {x, -8., 8.}]
Plot[Sum[an1[i] * Cos[kp[i] * x] , {i, 1, 7, 2}] + .5, {x, -8., 8.}]
Plot[Sum[an1[i] * Cos[kp[i] * x] , {i, 1, 25, 2}] + .5, {x, -8., 8.}]
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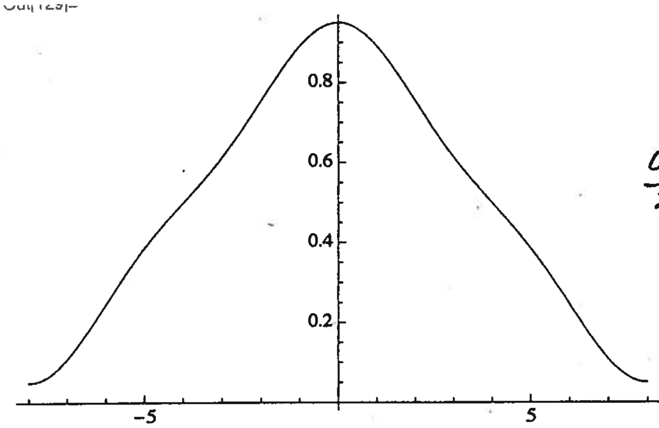
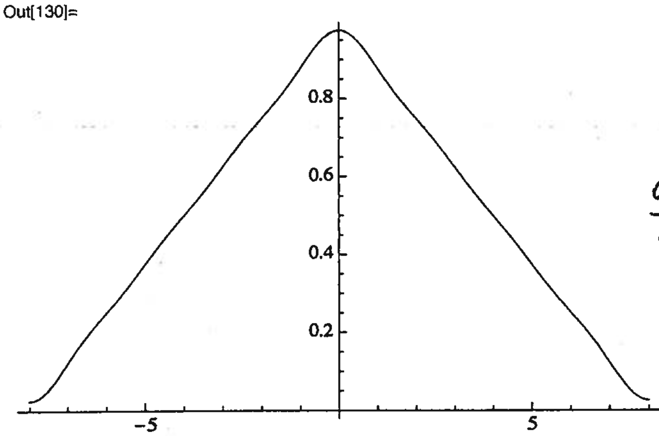
(Debug) Out[129]=



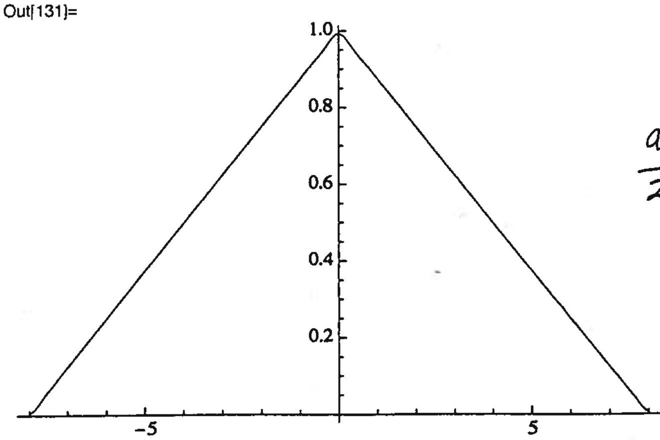
$$\frac{a_0}{2} + \sum_{n=1}^{3} a_n \cos\left(\frac{n\pi x}{L}\right) \qquad L=8$$

(Debug) Out[130]=



$$\frac{a_0}{2} + \sum_{n=1}^{7} a_n \cos\left(\frac{n\pi x}{L}\right)$$

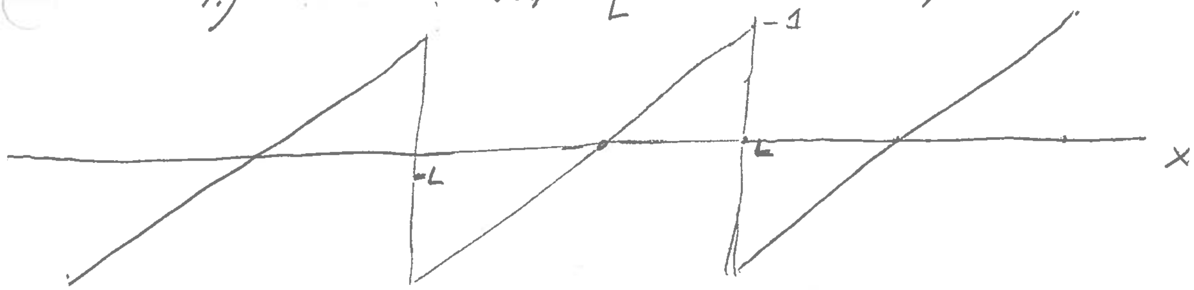
(Debug) Out[131]=



$$\frac{a_0}{2} + \sum_{n=1}^{25} a_n \cos\left(\frac{n\pi x}{L}\right)$$

HW 9 Due 4/22/09

1.) $f(x) = \frac{x}{L} \quad -L < x < L$, and $f(x+2L) = f(x)$ is periodic



Find the Fourier series

$$a_n = \frac{1}{L}\int_{-L}^{L} dx\, \frac{x}{L}\cos\left(\frac{n\pi x}{L}\right) = 0 \quad \rightarrow \text{by symmetry}$$

$$\frac{a_0}{2} = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} dx\, \frac{x}{L}\sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{L^2}\int u\,dv, \qquad u = x, \quad v = \frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{1}{L^2}\left(uv\Big|_{-L}^{L} - \int_{-L}^{L} v\,du\right)$$

$$= \frac{1}{L^2}\left(\frac{-Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{1}{L^2}\left(\frac{-L^2}{n\pi}\cos(n\pi) - \frac{L^2}{n\pi}\cos(-n\pi) + \underbrace{\frac{L}{n\pi}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}}_{\Downarrow\ 0}\right)$$

$$= -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

so $\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{L}\right)}$

```
sign = -1;
For[n = 1, n < max, n += 1,
  sign = -1 sign;
  an[n] = sign 2. / (n Pi);
  kp[n] = n Pi / length;
  kh[n] = kp[n] h;
 ];

Plot[an[1] Sin[kp[1] x], {x, -8., 8.}]
Plot[Sum[an[i] * Sin[kp[i] * x] , {i, 1, 3}], {x, -8., 8.}]
Plot[Sum[an[i] * Sin[kp[i] * x] , {i, 1, 5}], {x, -8., 8.}]
Plot[Sum[an[i] * Sin[kp[i] * x] , {i, 1, 9, 1}], {x, -8., 8.}]
Plot[Sum[an[i] * Sin[kp[i] * x] , {i, 1, 21, 1}], {x, -8., 8.}]
Plot[Sum[an[i] * Sin[kp[i] * x] , {i, 1, 99, 1}], {x, -8., 8.}]
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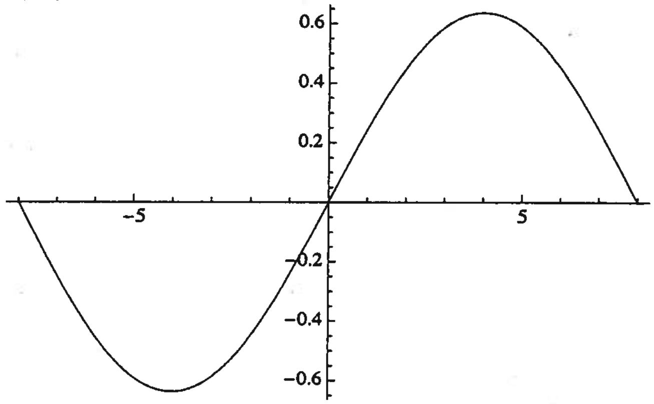
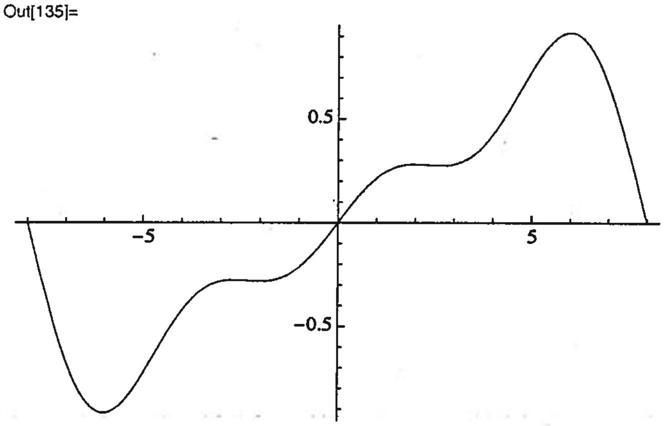


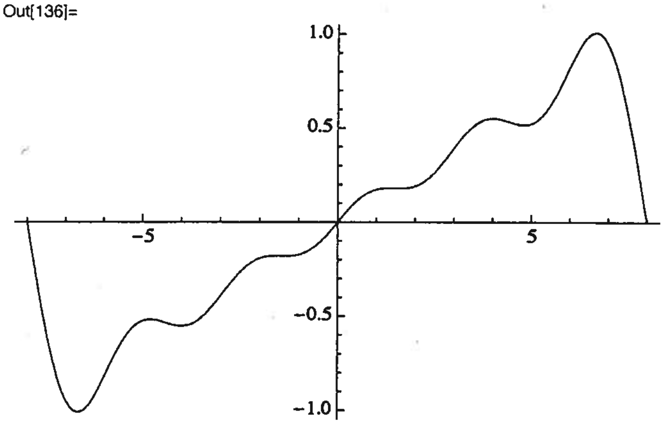
$b_1 \sin\left(\frac{n\pi x}{L}\right)$ $\quad L=8$

(Debug) Out[135]=

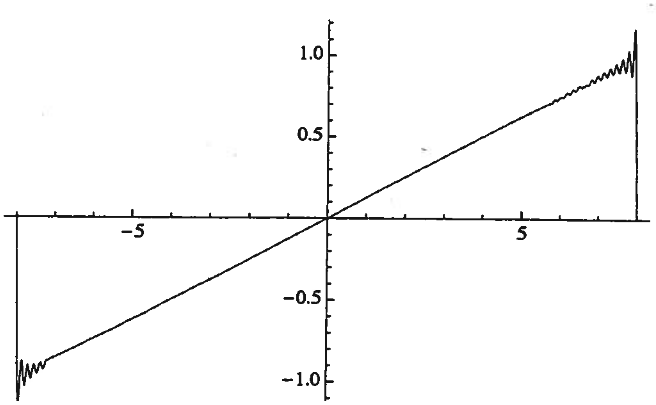


$\sum_{n\ \text{odd}}^{3} b_n \sin\left(\frac{n\pi x}{L}\right)$

(Debug) Out[136]=

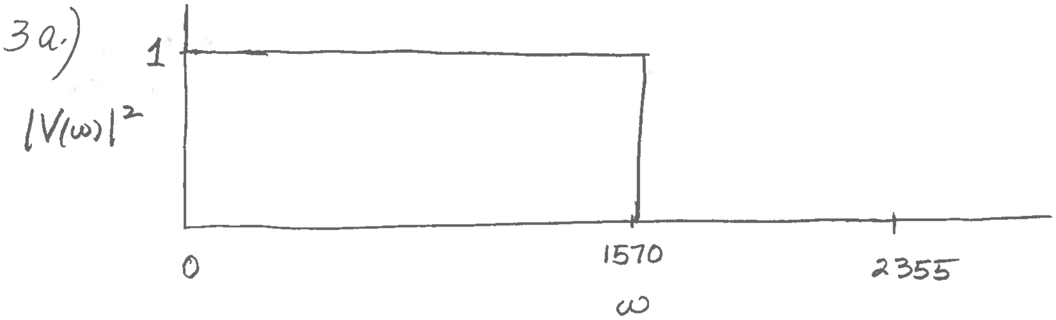


$\sum_{n\ \text{odd}}^{5} b_n \sin\left(\frac{n\pi x}{L}\right)$



$\sum_{n\ \text{odd}}^{25} b_n \sin\left(\frac{n\pi x}{L}\right)$

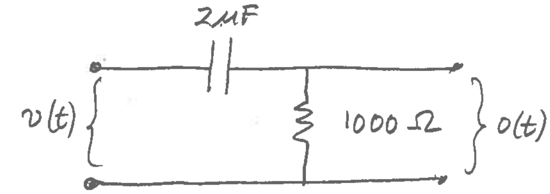
3a.)



$\omega_c = 2\pi(250) = 1570 \frac{\text{rad}}{\text{s}}$

$1.5\,\omega_c = 2355 \frac{\text{rad}}{\text{sec}}$

3b.)



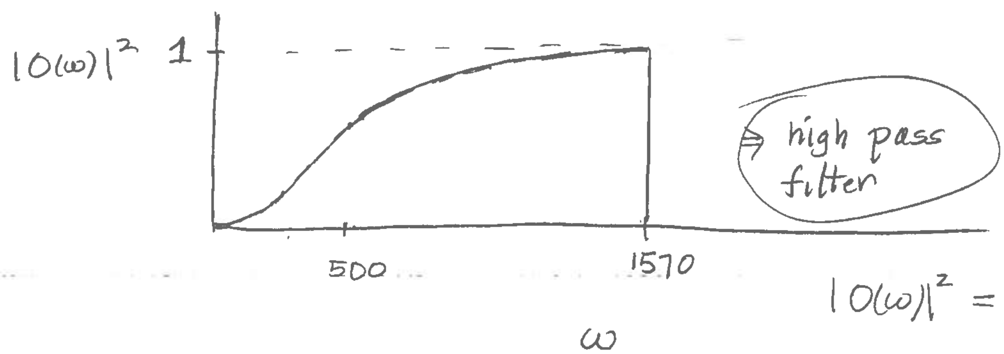
$$\left.\begin{aligned} V(\omega) - \frac{Q(\omega)}{C} - RI(\omega) &= 0 \\ O(\omega) &= RI(\omega) \\ -i\omega Q(\omega) &= I(\omega) \end{aligned}\right\} \Rightarrow \quad O(\omega) = \frac{R}{R + \frac{1}{-i\omega C}} V(\omega) = \frac{\omega V(\omega)}{\omega + \frac{i}{RC}}$$

$$|O(\omega)|^2 = \frac{\omega^2}{\omega^2 + (1/RC)^2} \qquad 0 < \omega < \omega_c$$

$RC = (1000)(2\times10^{-6})$

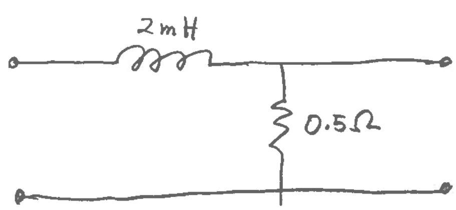
$= 2\times10^{-3}$

$\frac{1}{RC} = 500$



$$|O(\omega)|^2 = \frac{\omega^2}{\omega^2 + (500)^2}, \quad 0 < \omega < \omega_c$$

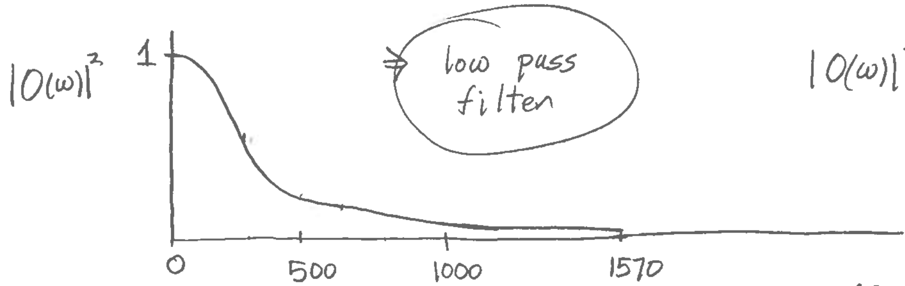
3c.)



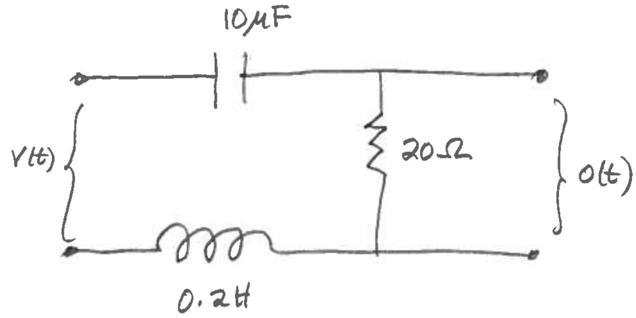
$$\left.\begin{aligned} V(\omega) - (-i\omega L)I(\omega) - RI(\omega) &= 0 \\ O(\omega) &= R\,I(\omega) \end{aligned}\right\} \quad O(\omega) = \frac{R\,V(\omega)}{R - i\omega L} = \frac{i\frac{R}{L}\,V(\omega)}{\omega + \frac{iR}{L}}$$

$$R/L = \frac{0.5}{2\times10^{-3}} = 250$$

$$|O(\omega)|^2 = \frac{(R/L)^2\,|V(\omega)|^2}{\omega^2 + (R/L)^2} = \frac{(250)^2}{\omega^2 + (250)^2}$$



3d.)



$$V(\omega) - \frac{I(\omega)}{-i\omega C} - RI(\omega) - L(-i\omega)I =$$

$$O(\omega) = R\,I(\omega)$$

$$\Rightarrow O(\omega) = \frac{R \quad V(\omega)}{R - i\omega L + \frac{1}{-i\omega C}} \times \left(\frac{\frac{i\omega}{L}}{\frac{i\omega}{L}}\right)$$

$$= \frac{i\omega \frac{R}{L} \quad V(\omega)}{\omega^2 + i\omega\frac{R}{L} - \frac{1}{LC}}$$

$$|O(\omega)|^2 = \frac{\omega^2\left(\frac{R}{L}\right)^2 \ |V(\omega)|^2}{\left(\omega^2 - \frac{1}{LC}\right)^2 + \omega^2\frac{R^2}{L^2}}$$

$$\frac{R}{L} = \frac{20}{0.2} = 100$$

$$\frac{1}{LC} = \frac{1}{(0.2)(10\times10^{-6})} = 0.5\times10^6 \simeq (707)^2$$

$$|O(\omega)|^2 = \frac{10^4\omega^2}{(\omega^2 - 707^2)^2 + 10^4\omega^2}, \quad 0<\omega<\omega_c$$