HW #8

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for eq. 15.3 — $\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx = 2L\left(\frac{1}{2}\right) = L$

for eq. 15.4 — $\frac{1}{2}\int_{-L}^{L} \cos\left[\frac{(n+m)\pi x}{L}\right] + \cos\left[\frac{(n-m)\pi x}{L}\right] dx$

$$= \frac{1}{2}\left[\frac{L}{(n+m)\pi} \sin\frac{(n+m)\pi x}{L}\Big|_{-L}^{L} + \frac{L}{(n-m)\pi} \sin\frac{(n-m)\pi x}{L}\Big|_{-L}^{L}\right]$$

$$= 0$$

(15.5) $= \frac{1}{2}\int \cos\left[\frac{(n-m)\pi x}{L}\right] - \cos\left[\frac{(n+m)\pi x}{L}\right] dx = 0$

(15.6) $= \frac{1}{2}\int \sin\left[\frac{(n+a)\pi x}{L}\right] + \sin\left[\frac{(n-m)\pi x}{L}\right] dx = 0$

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$$f(x) = \frac{1}{2}a_0 + \sum_n a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L} dx = \frac{1}{2}a_0 \int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right) + \sum_n a_n \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx + \sum_n b_n \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L} dx = a_m L$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} dx$$

$$\boxed{\int_{-L}^{L} f(x)dx = \int_{-L}^{L}\frac{a_0}{2}dx \qquad a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx}$$

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$$f(x) = \frac{1}{2}a_0 + \sum_n a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{2}a_0 + \sum_n\left[\frac{a_n}{2}e^{\frac{in\pi x}{L}} + \frac{a_n}{2}e^{-\frac{in\pi x}{L}} + \frac{b_n}{2i}e^{\frac{in\pi x}{L}} - \frac{b_n}{2i}e^{-\frac{in\pi x}{L}}\right]$$

$$= \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-\frac{in\pi x}{L}} + \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{\frac{in\pi x}{L}}$$

$$= \sum_{n=-\infty}^{-1}\frac{a_{|n|} + ib_{|n|}}{2}e^{\frac{in\pi x}{L}} + \frac{a_0}{2}e^{0\left(\frac{i\pi x}{L}\right)} + \sum_{n=1}^{\infty}\frac{a_n - ib_n}{2}e^{\frac{in\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \text{ where } 2c_n = a_{|n|} + ib_{|n|} \text{ for } n<0$$

$c_n = 0$ for $n = 0$

$2c_n = a_n - ib_n$ for $n > 0$

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for $n>0$

$$C_n = \frac{a_n - i b_n}{2} = \frac{1}{2L}\int f(x)\,dx\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right] = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}}\,dx$$

for $n=0$

$$C_n = \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} f(x) e^{0\left(\frac{\pi x}{L}\right)}\,dx$$

$n<0$

$$C_n = \frac{a_{|n|} + i b_{|n|}}{2} = \frac{1}{2L}\int f(x)\,dx\left[\cos\left(\frac{|n|\pi x}{L}\right) + i\sin\left(\frac{|n|\pi x}{L}\right)\right]$$

$$= \frac{1}{2L}\int f(x)\,dx\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= \frac{1}{2L}\int f(x) e^{-\frac{in\pi x}{L}}\,dx$$

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for finite interval — $f(x) = \sum_n C_n e^{\frac{in\pi x}{L}}$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}}\,dx$$

Let $k = \frac{n\pi}{L}$

$$e^{\frac{in\pi x}{L}} \rightarrow e^{ikx}$$

$$f(x) = \int_{-\infty}^{\infty} dk\, F(k) e^{ikx}$$

$$F(k) = \frac{L}{\pi} C_n = \frac{1}{2\pi}\int_{-L}^{L} f(x) e^{-ikx}\,dx$$

224a-d

$F(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,\delta(x-x_0)\,dx = \frac{1}{2\pi}e^{-ikx_0}$ a.

for $x_0=0$, $F(k)=\frac{1}{2\pi}$ b.

$f(x)=\int_{-\infty}^{\infty}F(k)e^{ikx}dk$ e

$\delta(x-x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik(x-x_0)}dk$

$F(k)=\delta(k-k_0)$

$f(x)=\int_{-\infty}^{\infty}\delta(k-k_0)e^{ikx}dk=e^{ik_0x}$

$F(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$

$\delta(k-k_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i(k-k_0)x}dx$

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$f(x)=\int_{-\infty}^{\infty}F(k)e^{ikx}dk = c\int_{-\infty}^{\infty}\hat{F}(k)e^{ikx}dk$

$F(k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$

$c\tilde{F}(k)=$ " "

$\hat{F}(k)=\frac{1}{2\pi c}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$

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$$\boxed{f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx} dk = -\int_{\infty}^{-\infty} F(-k) e^{-ikx} dk = \int_{-\infty}^{\infty} \hat{F}(k) e^{-ikx} dk}$$

$$\boxed{\hat{F}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx} dx}$$

228 a-d

$$\boxed{f(t) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega} \qquad \boxed{h(t) = \int_{-\infty}^{\infty} H(\omega) e^{-i\omega t} d\omega}$$

a.

$$F(\omega) H(\omega) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt\right]\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} h(t) e^{i\omega t} dt\right]$$
$$= \frac{1}{(2\pi)^2} \iint f(t_1) h(t_2) e^{i\omega(t_1+t_2)} dt_1 dt_2$$

b.-d.

$$\int_{-\infty}^{\infty} F(\omega) H(\omega) e^{-i\omega t} d\omega = \frac{1}{(2\pi)^2} \iiint f(t_1) h(t_2) e^{i\omega(t_1+t_2-t)} dt_1 dt_2 d\omega$$
$$= \frac{1}{(2\pi)^2} \iint f(t_1) h(t_2) \delta(t_1+t_2-t) dt_1 dt_2 \, (2\pi)$$
$$= \frac{1}{2\pi} \int dt_2 \, f(t-t_2) h(t_2) \, dt_2$$
$$= \frac{1}{2\pi} \int d\tau \, f(t-\tau) h(\tau) \, d\tau$$

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$$\int_{-\infty}^{\infty} F(\omega) H^*(\omega) e^{-i\omega t} d\omega = \frac{1}{(2\pi)^2} \iiint f(t_1) h^*(t_2) e^{i\omega(t_1-t_2-t)} dt_1 dt d\omega$$
$$= \frac{1}{2\pi} \int dt_2 \, f(t+t_2) h^*(t_2)$$

Now if $t=0$ and $h(t) = f(t)$ [which means $H(\omega) = F(\omega)$]

$$\int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega = \frac{1}{2\pi} \int d\tau \, f(\tau) f^*(\tau)$$

$$\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} \int |f(t)|^2 dt$$