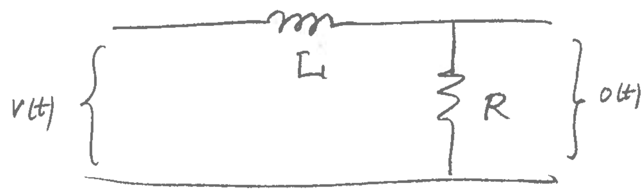
Solutions for HW 11

Green's fn. for RL circuit is response to $\delta(t)$ input



$$V(\omega) - (-i\omega)L\, I(\omega) - R I(\omega) = 0$$

$$O(\omega) = R\, I(\omega)$$

$$\Rightarrow O(\omega) = \frac{R V(\omega)}{R - i\omega L} \times \frac{\frac{i}{L}}{\frac{i}{L}}$$

$$= \frac{\frac{iR}{L} V(\omega)}{\omega + \frac{iR}{L}}$$

$$= \left[\frac{\frac{iR}{L}\frac{1}{2\pi}}{\omega + iR/L}\right]\left[2\pi V(\omega)\right]$$

If $V(\omega) = \frac{1}{2\pi}$, then

$$V(t) = \int d\omega\, e^{-i\omega t} V(\omega) = \frac{1}{2\pi}\int d\omega\, e^{i\omega t} = \delta(t)$$

$\rightarrow V(\omega) = \frac{1}{2\pi}$, would give $O(t) = g(t)$, the Green's fn.

That means that

$$\boxed{G(\omega) = \frac{iR/L\ \frac{1}{2\pi}}{\omega + iR/L}}\ , \text{ which is } O(\omega) \text{ when } V(\omega) = \frac{1}{2\pi}$$

$$g(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \frac{i\frac{R}{L}\frac{1}{2\pi}}{\omega + \frac{iR}{L}}$$

$$\rightarrow \int dz\, e^{-izt}\, \frac{\frac{iR}{L}\frac{1}{2\pi}}{z + \frac{iR}{L}}$$

pole at $z = -\frac{iR}{L}$

$$e^{-izt} = e^{-ixt} e^{yt} \longrightarrow 0 \quad \text{when } yt \to -\infty$$

if $t > 0$, $y \xrightarrow{\text{need}} -\infty \rightarrow$ use C_-

if $t < 0$, $y \xrightarrow{\text{need}} +\infty \rightarrow$ use C_+

$$g(t) = \frac{iR}{L}\frac{1}{2\pi}\left[\theta(t)\int_{C-} dz\, \frac{e^{-izt}}{z+\frac{iR}{L}} + \theta(-t)\int_{C+} dz\, \frac{e^{-izt}}{z+\frac{iR}{L}}\right]$$

$$\downarrow \qquad\qquad \downarrow$$

$$-2\pi i \ \mathrm{Res}\left(z=-\frac{iR}{L}\right) \qquad\qquad 0$$

$$= \frac{iR}{L}\frac{1}{2\pi}(-2\pi i)\ e^{-izt}\Big|_{z=-\frac{iR}{L}}$$

$$\boxed{g(t) = \theta(t)\frac{R}{L}e^{-R/L\,t}}$$

This is the Green's function, it vanishes for $t<0$, which expresses causality [no output before the input that causes it].

$$V(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\frac{\omega_c^2}{\omega^2+\omega_c^2}$$

$$= \int_{-\infty}^{\infty} dz\, e^{-izt}\frac{\omega_c^2}{(z+i\omega_c)(z-i\omega_c)}$$

as before $t>0$, use $C-$

$t<0$ use $C+$

$$V(t) = \theta(t)\int_{C_-} dz\, \frac{e^{-izt}\omega_c^2}{(z+i\omega_c)(z-i\omega_c)} + \theta(-t)\int_{C_+} dz\, \frac{e^{-izt}\omega_c^2}{(z+i\omega_c)(z-i\omega_c)}$$

$$= \theta(t)(-2\pi i)\, e^{-izt}\frac{\omega_c^2}{z-i\omega_c}\Big|_{z=-i\omega_c} + \theta(-t)(2\pi i)\frac{e^{-izt}\omega_c^2}{z+i\omega_c}\Big|_{z=i\omega_c}$$

$$= \theta(t)\,\pi\omega_c e^{-\omega_c t} + \theta(-t)\,\pi\omega_c e^{\omega_c t}$$

$$\boxed{V(t) = \pi\omega_c e^{-\omega_c|t|}}$$

Convolution to obtain output $O(t)$

$$O(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\ g(t-\tau)\left[2\pi V(\tau)\right]$$

Note: $\frac{1}{2\pi} \times 2\pi V(\tau)$ in Snieder's convention, and the 2π factors cancel here

$$= \int_{-\infty}^{\infty} d\tau\ \underbrace{\theta(t-\tau)\,\frac{R}{L}\,e^{-R/L\,(t-\tau)}}_{g(t-\tau)}\ \underbrace{\pi\omega_c\,e^{-\omega_c|\tau|}}_{V(\tau)}$$

$$= \frac{\pi\omega_c R}{L}\int_{-\infty}^{t} d\tau\ e^{-R/L\,(t-\tau)}\,e^{-\omega_c|\tau|}$$

$\rightarrow$ only way to do the integral is to split the range to $\underset{|\tau|=-\tau}{\int_{-\infty}^{0} d\tau\ e^{-\omega_c\tau}} + \underset{|\tau|=\tau}{\int_{0}^{t} d\tau\ e^{\omega_c\tau}}$

$$O(t) = \frac{\pi\omega_c R}{L}\left[\int_{-\infty}^{0} d\tau\ e^{-R/L\,(t-\tau)}\,e^{+\omega_c\tau} + \int_{0}^{t} d\tau\ e^{-R/L\,(t-\tau)}\,e^{-\omega_c\tau}\right]$$

$$= \frac{\pi\omega_c R}{L}\,e^{-R/L\,t}\left[\int_{-\infty}^{0} d\tau\ e^{(\omega_c+R/L)\tau} + \int_{0}^{t} d\tau\ e^{-(\omega_c-R/L)\tau}\right]$$

$\downarrow$

$$\left[\frac{1}{\omega_c+R/L}\,e^{(\omega_c+R/L)\tau}\Big|_{-\infty}^{0} + \frac{-1}{\omega_c-\frac{R}{L}}\,e^{-(\omega_c-R/L)\tau}\Big|_{0}^{t}\right]$$

$$\left[\frac{1}{\omega_c+R/L} - \frac{1}{\omega_c-R/L}\left(e^{-(\omega_c-R/L)t} - 1\right)\right]$$

$$\left[\underbrace{\frac{1}{\omega_c+R/L} + \frac{1}{\omega_c-\frac{R}{L}}}_{\frac{2\omega_c}{\omega_c^2-\frac{R^2}{L^2}}} - \frac{1}{\omega_c-\frac{R}{L}}\,e^{-(\omega_c-R/L)t}\right]$$

This gives

$$O(t) = \frac{\pi \omega_c R}{L} e^{-R/L\, t} \left[\frac{2\omega_c}{\omega_c^2 - \frac{R^2}{L^2}} - \frac{1}{\omega_c - \frac{R}{L}} e^{-(\omega_c - R/L)t} \right]$$

$$\boxed{O(t) = 2\pi \frac{R}{L} \frac{\omega_c^2}{\omega_c^2 - \frac{R^2}{L^2}} e^{-R/L\, t} - \frac{\pi R/L\, \omega_c}{\omega_c - R/L} e^{-\omega_c t}}$$

2.) Newton for spring-mass-damper system.

a.) $$m \frac{d^3 x}{dt^2} = -kx - b\frac{dx}{dt} + f(t)$$

can be written as

$$\left(m \frac{d^2}{dt^2} + b\frac{d}{dt} + k\right) x(t) = f(t)$$

Let $D(y) = my^2 + by + k$

then $\underline{D(\frac{d}{dt})\, x(t) = f(t)}$

b.) For $x(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} X(\omega)$, $f(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} F(\omega)$

$$D(\tfrac{d}{dt})\, x(t) = \int_{-\infty}^{\infty} d\omega\, D(\tfrac{d}{dt})\, e^{-i\omega t} X(\omega), \quad \text{use } \frac{d}{dt} e^{-i\omega t} = -i\omega e^{-i\omega t}$$

$$= \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} D(-i\omega) X(\omega) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} F(\omega)$$

linear independence of $e^{-i\omega t}$ for different ω values means that this relation must be true for each ω,

i.e., $$D(-i\omega) X(\omega) = F(\omega)$$

so $$X(\omega) = \left[\frac{1/2\pi}{D(-i\omega)}\right] \left[2\pi F(\omega)\right]$$

c.) given $\quad g(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, G(\omega)$

$$D\left(\tfrac{d}{dt}\right) g(t) = \int_{-\infty}^{\infty} d\omega\; D\left(\tfrac{d}{dt}\right) e^{-i\omega t}\, G(\omega)$$

$$= \int_{-\infty}^{\infty} d\omega\; e^{-i\omega t}\, \underbrace{D(-i\omega)\, G(\omega)}_{=1/2\pi}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\; e^{-i\omega t}$$

$$= \delta(t)$$

so $\underline{D\left(\tfrac{d}{dt}\right) g(t) = \delta(t)}$

d.) $$g(t) = \int_{-\infty}^{\infty} d\omega\; e^{-i\omega t} \underbrace{\left[\frac{1/2\pi}{-m\omega^2 - ib\omega + k}\right]}_{G(\omega)}$$

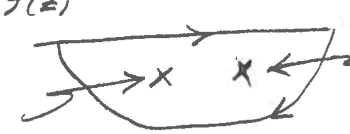
$$= \frac{-1}{2\pi m} \int d\omega\; e^{-i\omega t} \frac{1}{\left(\omega^2 + \frac{ib}{m}\omega - \frac{k}{m}\right)}$$

$$= \left(\omega + \frac{ib}{2m}\right)^2 - \frac{k}{m} + \frac{b^2}{4m^2}$$

Let $\omega_d^2 \equiv \frac{k}{m} - \frac{b^2}{4m^2}$, so it is $\left(\omega + \frac{ib}{2m}\right)^2 - \omega_d^2$

$$g(t) = -\frac{1}{2\pi m} \int_{-\infty}^{\infty} dz\; e^{-izt} \underbrace{\frac{1}{\left(z + \frac{ib}{2m} + \omega_d\right)\left(z + \frac{ib}{2m} - \omega_d\right)}}_{f(z)}$$

For $t > 0$, use $C_- =$



$-\frac{ib}{2m} + \omega_d$

$-\frac{ib}{2m} - \omega_d$

$$g(t) = -2\pi i \left(\frac{-1}{2\pi m}\right)\left(\mathrm{Res}\left(z = -\frac{ib}{2m} - \omega_d\right) + \mathrm{Res}\left(z = -\frac{ib}{2m} + \omega_d\right)\right)$$

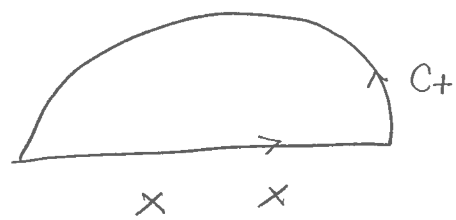
$$\text{Res}\left(z = \frac{-ib}{2m} - \omega_d\right) = \left.\frac{e^{-izt}}{z + \frac{ib}{2m} - \omega_d}\right|_{z = \frac{-ib}{2m} - \omega_d} = \frac{e^{-\frac{bt}{2m}} e^{i\omega_d t}}{-2\omega_d}$$

$$\text{Res}\left(z = -\frac{ib}{2m} + \omega_d\right) = \left.\frac{e^{-izt}}{z + \frac{ib}{2m} + \omega_d}\right|_{z = -\frac{ib}{2m} + \omega_d} = \frac{e^{-\frac{bt}{2m}} e^{-i\omega_d t}}{2\omega_d}$$

so $$g(t) = \frac{-i}{2m\omega_d} e^{-\frac{bt}{2m}} \left(e^{i\omega_d t} - e^{-i\omega_d t}\right)$$

$$g(t) = \frac{1}{m\omega_d} e^{-bt/2m} \sin(\omega_d t) \quad \text{— } \underline{\text{for } t > 0}$$

For $t < 0$, contour C_+ must be chosen



$g(t) = 0$ for $t<0$ because there are no poles in upper half plane.

so $$\boxed{g(t) = \frac{\theta(t)}{m\omega_d} e^{-bt/2m} \sin(\omega_d t)}$$ where $\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$ is the step function.

3.) a.) For $V(\vec{r}) = \delta^{(3)}(\vec{r})$

$$\tilde{V}(\vec{k}) = \int d^3r\, e^{i\vec{k}\cdot\vec{r}}\, \delta^{(3)}(\vec{r}) = \underline{\underline{1}}$$

b.) $$(\nabla^2 - \mu^2)V(r) = \frac{1}{(2\pi)^3}\int d^3k (\nabla^2 - \mu^2) e^{-i\vec{k}\cdot\vec{r}}\, \tilde{V}(\vec{k})$$

$$\nabla^2 e^{-i\vec{k}\cdot\vec{r}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) e^{-ik_x x - ik_y y}\, e^{-ik_z z}$$
$$= (-k_x^2 - k_y^2 - k_z^2)\, e^{-i\vec{k}\cdot\vec{r}}$$
$$= -\vec{k}^2 e^{-i\vec{k}\cdot\vec{r}}$$

$$\therefore (\nabla^2 - \mu^2)V(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-i\vec{k}\cdot\vec{r}}\left[(-\vec{k}^2 - \mu^2)\tilde{V}(\vec{k})\right] = \frac{1}{(2\pi)^3}\int d^3k\, e^{-i\vec{k}\cdot\vec{r}}\left[\frac{-\tilde{\rho}(\vec{k})}{\epsilon_0}\right]$$

Because of linear independence of $e^{-i\vec{k}\cdot\vec{r}}$ for different $\vec{k}$ values, it is necessary that

$$(-\vec{k}^2 - \mu^2)\, \tilde{V}(\vec{k}) = -\frac{\tilde{\rho}(\vec{k})}{\epsilon_0}$$

so $$\tilde{V}(\vec{k}) = \frac{-1}{\vec{k}^2 + \mu^2}\left[-\frac{\tilde{\rho}(\vec{k})}{\epsilon_0}\right].$$

c.) $$G(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k\, e^{-i\vec{k}\cdot\vec{r}}\, \frac{-1}{\vec{k}^2 + \mu^2}$$

choose $\vec{r}$ along $\hat{e}_z$
so $\vec{k}\cdot\vec{r} = kr\cos\theta$

$$= \frac{-1}{(2\pi)^3}\int_0^\infty dk\, k^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta\, e^{-ikr\cos\theta}\, \frac{1}{k^2 + \mu^2}$$

Here $k = |\vec{k}|$, $r = |\vec{r}|$.

Let $x = \cos\theta$, $dx = -\sin\theta\, d\theta$

$$\int_0^\pi d\theta \sin\theta\, e^{-ikr\cos\theta} = \int_1^{-1} -dx\, e^{-ikrx} = \int_{-1}^{1} dx\, e^{-ikrx}.$$

The angle integral $\int_0^{2\pi} d\phi$ gives overall factor 2π

the integral $\int_{-1}^{1} dx\, e^{-ikrx} = \left.\dfrac{e^{-ikrx}}{-ikr}\right|_{-1}^{1} = \dfrac{e^{ikr} - e^{-ikr}}{ikr}$

so $G(\vec{r}) = \dfrac{-1}{(2\pi)^2} \displaystyle\int_0^\infty dk\, k^2 \, \frac{e^{ikr} - e^{-ikr}}{ikr} \, \frac{1}{k^2+\mu^2}$

$$= \frac{-1}{(2\pi)^2 ir} \int_0^\infty dk\, k \left(e^{ikr} - e^{-ikr}\right) \frac{1}{k^2+\mu^2}$$

the part involving e^{-ikr} is transformed as follows

$$\int_0^\infty dk\,k \; -e^{-ikr} \frac{1}{k^2+\mu^2} \xrightarrow[k'=-k]{} -\int_0^{-\infty} dk'\,k' \, e^{ik'r} \frac{1}{k'^2+\mu^2}$$

$$= \int_{-\infty}^{0} dk'\,k' \, e^{ik'r} \frac{1}{k'^2+\mu^2}$$

— it is the same as the part involving e^{ikr} except for the limits now being $-\infty$ to 0. The two parts add up to

$$G(r) = \frac{-1}{(2\pi)^2 ir} \int_{-\infty}^{\infty} dk\,k\, e^{ikr} \frac{1}{k^2+\mu^2}$$

$r = |\vec{r}|$ is positive, so close contour in C_+ - upper ½ plane

$$G(r) = \frac{-1}{(2\pi)^2 ir} \int_{C_+} dz\, z\, e^{izr} \frac{1}{(z+i\mu)(z-i\mu)}$$

$$= \frac{-1}{(2\pi)^2 ir} (2\pi i) \operatorname{Res}(z=i\mu)$$

$$G(r) = -\frac{1}{2\pi r} \left.\frac{z}{z+i\mu} e^{izr}\right|_{z=i\mu} = \boxed{-\frac{1}{4\pi r} e^{-\mu r} = G(r)}$$