Derivation of Compton's Equation

Let λ_1 and λ_2 be the wavelengths of the incident and scattered x rays, respectively, as shown in Figure 3-18. The corresponding momenta are

$$p_1 = \frac{E_1}{c} = \frac{hf_1}{c} = \frac{h}{\lambda_1}$$

and

$$p_2 = \frac{E_2}{c} = \frac{hf_2}{c} = \frac{h}{\lambda_2}$$

using $f\lambda = c$. Since Compton used the K_α line of molybdenum ($\lambda = 0.0711$ nm; see Figure 3-15*b*), the energy of the incident x ray (17.4 keV) is much greater than the binding energy of the valence electrons in the carbon-scattering block (about 11 eV); therefore, the carbon electrons can be considered to be free.

Conservation of momentum gives

$$\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_e$$

or

$$\begin{aligned} p_e^2 &= p_1^2 + p_2^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 \\ &= p_1^2 + p_2^2 - 2p_1 p_2 \cos\theta \qquad \mathbf{3\text{-}26} \end{aligned}$$

where $\boldsymbol{p}_e$ is the momentum of the electron after the collision and θ is the scattering angle of the photon, measured as shown in Figure 3-18. The energy of the electron before the collision is simply its rest energy $E_0 = mc^2$ (see Chapter 2). After the collision, the energy of the electron is $(E_0^2 + p_e^2c^2)^{1/2}$.

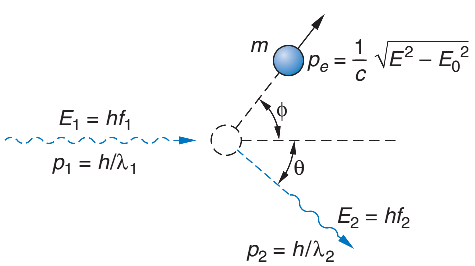

FIGURE 3-18 The scattering of x rays can be treated as a collision of a photon of initial momentum h/λ_1 and a free electron. Using conservation of momentum and energy, the momentum of the scattered photon h/λ_2 can be related to the initial momentum, the electron mass, and the scattering angle. The resulting Compton equation for the change in the wavelength of the x ray is Equation 3-25.

Conservation of energy gives

$$p_1c + E_0 = p_2c + (E_0^2 + p_e^2c^2)^{1/2}$$

Transposing the term p_2c and squaring, we obtain

$$E_0^2 + c^2(p_1 - p_2)^2 + 2cE_0(p_1 - p_2) = E_0^2 + p_e^2c^2$$

or

$$p_e^2 = p_1^2 + p_2^2 - 2p_1p_2 + \frac{2E_0(p_1 - p_2)}{c} \qquad \textbf{3-27}$$

Eliminating p_e^2 between Equations 3-26 and 3-27, we obtain

$$\frac{E_0(p_1 - p_2)}{c} = p_1p_2(1 - \cos\theta)$$

Multiplying each term by $hc/p_1p_2E_0$ and using $\lambda = h/p$, we obtain *Compton's equation*:

$$\lambda_2 - \lambda_1 = \frac{hc}{E_0}(1 - \cos\theta) = \frac{hc}{mc^2}(1 - \cos\theta)$$

or

$$\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos\theta) \qquad \textbf{3-25}$$