

Lecture #8 Summary

PHYS 313 Electricity and Magnetism I

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Fall 2026

Poisson's Equation and the Potential Function

The (static) electric field has the following properties:

$$(1) \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad (2) \vec{\nabla} \times \vec{E} = 0$$

Now we can substitute in $\vec{E} = -\vec{\nabla}V$ using the potential function V :

$$(1) \vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla}V) = -\nabla^2 V = \rho/\epsilon_0 \quad \text{Poisson's Equation, a scalar equation for } V$$

$$(2) \vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla}V) = 0 \text{ always (nothing new here)}$$

What are the advantages of using the potential function to find $\vec{E}$ instead of calculating $\vec{E}$ directly?

(1) Solving $\nabla^2 V = -\rho/\epsilon_0$ is easier. There is only one differential equation to solve, rather than 2 vector equations ($\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ and $\vec{\nabla} \times \vec{E} = 0$).

How can the potential $V(x, y, z)$, a single scalar function of position, have the same information as the electric field $\vec{E}(x, y, z)$, a vector function of position? The vector actually has 3 scalar functions of position.

The components of electric field are inter-related through $\vec{\nabla} \times \vec{E} = 0$:

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \hat{y} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + \hat{z} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = 0$$

So

$$\frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_z}{\partial x} = \frac{\partial E_x}{\partial z}, \quad \frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}$$

So the pieces of $\vec{E}$ are strongly connected and constrained.

Also, $\vec{\nabla} \cdot \vec{E}$ is constrained to reproduce the charge density ρ that gave rise to the electric field. Hence $\vec{E}$ is very highly constrained. As a result, this information in vector $\vec{E}$ can be boiled down to a scalar function $V(x, y, z)$.

Now, can we use the potential function V to find the electric field of a charge distribution more easily? Yes.

$$\vec{E} = -\vec{\nabla}V$$

So, how do we express V in terms of a charge distribution $\rho(\vec{r}')$?

Start with a single point charge at the origin:

$$V(\vec{r}) = - \int_{\text{Ref. pt.}}^{\vec{r}} \vec{E} \cdot d\vec{l}$$



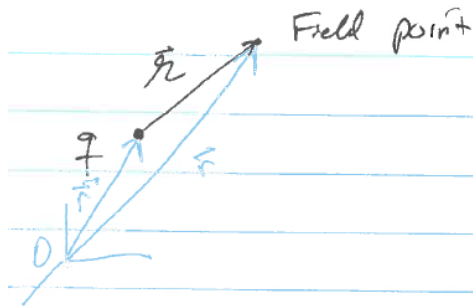
$$\vec{E}(\vec{r}') = \frac{q}{4\pi\epsilon_0 r'^2} \hat{r}$$

Only the radial component counts: $\hat{r} \cdot d\vec{l} = dr$

$$V(r) = - \int_{\infty}^r \frac{q}{4\pi\epsilon_0 r'^2} dr' = \frac{q}{4\pi\epsilon_0} \frac{1}{r'} \Big|_{\infty}^r = \frac{q}{4\pi\epsilon_0 r}$$

(Here we take the reference point to be the point at ∞ .)

A point charge has a potential that falls off like $1/r$.



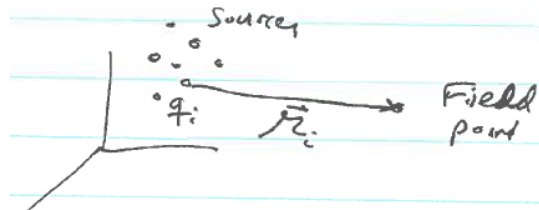
The more general expression moves the charge to a general source point $\vec{r}'$:

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 |\vec{z}|} \quad (\text{Ref. point at } \infty)$$

where $\vec{z} = \vec{r} - \vec{r}'$.

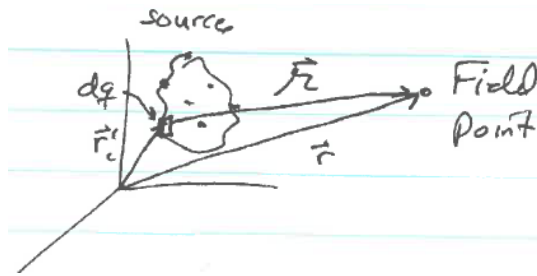
Using the superposition principle to calculate the total potential at a field point, one has

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \frac{q_i}{|\vec{z}_i|}$$



For a continuous distribution, this becomes

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{|\vec{z}|}$$



$$dq = \rho d\tau' \text{ in 3D} \quad \text{or} \quad dq = \sigma da' \text{ in 2D} \quad \text{or} \quad dq = \lambda dl' \text{ in 1D}$$

so

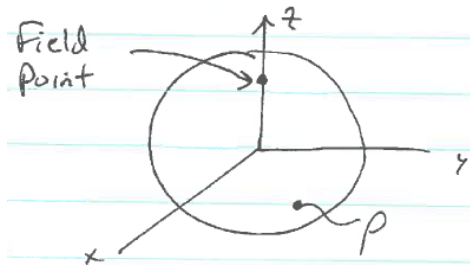
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{Volume}} \frac{\rho(\vec{r}') d\tau'}{|\vec{z}|} \quad \text{or} \quad \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\vec{r}') da'}{|\vec{z}|} \quad \text{or} \quad \frac{1}{4\pi\epsilon_0} \int_{\text{Line}} \frac{\lambda(\vec{r}') dl'}{|\vec{z}|}$$

Remember that these expressions all assume that the reference point for the potential is at infinity.

Do an example: Problem 2.29

Calculate the potential everywhere inside a uniformly charged solid sphere of radius R and total charge q .

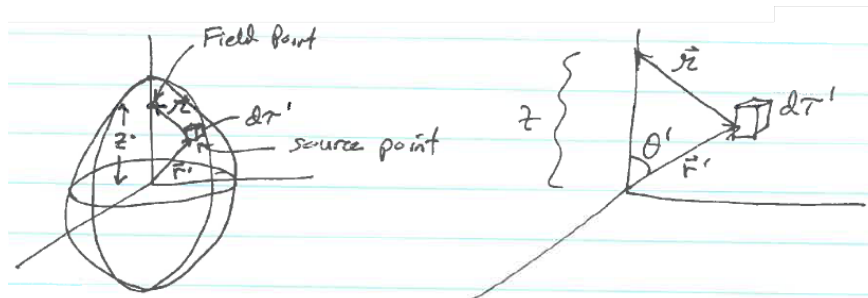
To simplify the math, put the z -axis of the spherical coordinate system through the required field point. Of course the origin will be chosen to be the center of the sphere!



Use the general equation for the volume source charge density:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}') d\tau'}{|\vec{z}|}$$

Now the integral is simple compared to the $\vec{E}$ integral expression: there is no vector quantity (like $\hat{r}$) to worry about, and there is one fewer power of $|\vec{z}|$ in the denominator!

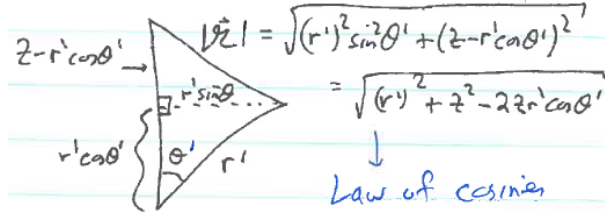


In this case,

$$\rho(\vec{r}') = \rho \text{ a constant,} \quad d\tau' = (r')^2 dr' \sin \theta' d\theta' d\phi'$$

From the geometry of the triangle in the right part of the figure above, we can find an expression for the magnitude of the Griffiths relative distance $|\vec{z}|$ using the law of cosines:

$$|\vec{z}| = \sqrt{(r')^2 \sin^2 \theta' + (z - r' \cos \theta')^2} = \sqrt{(r')^2 + z^2 - 2zr' \cos \theta'} \quad (\text{Law of cosines})$$



So the integral is

$$V(z) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \int_0^R \frac{\rho (r')^2 dr' \sin \theta' d\theta' d\phi'}{\sqrt{(r')^2 + z^2 - 2zr' \cos \theta'}}$$

(1) The ϕ' integral is easy since there is no ϕ' dependence in the integrand:

$$\int_0^{2\pi} d\phi' = 2\pi$$

(2) Next do the θ' integral (it is easier to do this one first).

The denominator is $a - bu$ where

$$a = (r')^2 + z^2, \quad b = 2zr', \quad u = \cos \theta' \Rightarrow du = -\sin \theta' d\theta'$$

$$\begin{aligned} \int_0^\pi \frac{\sin \theta' d\theta'}{\sqrt{a - bu}} &= - \int \frac{du}{\sqrt{a - bu}} = \frac{2}{b} (a - bu)^{1/2} \\ &= \frac{1}{zr'} \sqrt{(r')^2 + z^2 - 2zr' \cos \theta'} \Big|_0^\pi \\ &= \frac{1}{zr'} \left\{ \sqrt{(r')^2 + z^2 + 2zr'} - \sqrt{(r')^2 + z^2 - 2zr'} \right\} = \frac{1}{zr'} \{|r' + z| - |r' - z|\} \\ &= \begin{cases} 2/z & r' < z \\ 2/r' & r' > z \end{cases} \quad (\text{Take } z > 0) \end{aligned}$$

The result is

$$V(z) = \frac{1}{4\pi\epsilon_0} \rho 2\pi \int_0^R (r')^2 dr' \begin{cases} 2/z & r' < z \\ 2/r' & r' > z \end{cases}$$

(3) The r' integral has to be split into two parts:

$$\begin{aligned} V(z) &= \frac{2\pi\rho}{4\pi\epsilon_0} \left\{ 2 \int_0^z \frac{(r')^2 dr'}{z} + 2 \int_z^R r' dr' \right\} \\ &= \frac{4\pi\rho}{4\pi\epsilon_0} \left\{ \frac{1}{z} \cdot \frac{1}{3} (r')^3 \Big|_0^z + \frac{1}{2} (r')^2 \Big|_z^R \right\} \\ &= \frac{4\pi\rho}{4\pi\epsilon_0} \left\{ \frac{z^2}{3} - 0 + \frac{1}{2} (R^2 - z^2) \right\} \quad \left(\frac{2}{3} - \frac{2}{2} = -\frac{1}{3} \right) \\ V(z) &= \frac{4\pi\rho}{4\pi\epsilon_0} \cdot \frac{1}{2} \left\{ R^2 - \frac{z^2}{3} \right\} \end{aligned}$$

But the charge density can be written as $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, so

$$V(z) = \frac{q}{2\epsilon_0} \cdot \frac{3}{4\pi R^3} \left\{ R^2 - \frac{z^2}{3} \right\} = \frac{q}{8\pi\epsilon_0 R} \left(3 - \frac{z^2}{R^2} \right)$$

In general, we could orient the z -axis any way we want, so

$$V(r) = \frac{q}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right)$$

This is finite everywhere! It should go like $1/r$ outside the sphere: $V(r) = \frac{q}{4\pi\epsilon_0 r}$. Note that $V(r)$ and $\vec{\nabla}V \sim \partial V/\partial r$ is continuous at $r = R$.

$$\begin{aligned} \vec{E} = -\vec{\nabla}V &= - \left[\frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\phi} \right] \\ &= - \frac{(-2r)q}{8\pi\epsilon_0 R^3} \hat{r} \\ \vec{E} &= \frac{2q}{8\pi\epsilon_0} \frac{r}{R^3} \hat{r} \end{aligned}$$

At $r = R$, this becomes

$$\vec{E} = \frac{2q}{4\pi\epsilon_0 R^2} \hat{r}$$

Question: What happens at $|\vec{z}| = 0$ in $V(\vec{r}) \sim \int \frac{dq}{|\vec{z}|}$?

In other words, this is when the field point and source point coincide. Griffiths' argument: There is no charge at a point, so there is no contribution to V from that singular point.

Let's do some double-checks...

Now calculate $\vec{\nabla} \cdot \vec{E}$ and see if it equals ρ/ϵ_0 :

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{q}{4\pi\epsilon_0} \frac{r}{R^3} \right) + 0 + 0 = \frac{1}{r^2} \cdot \frac{3qr^2}{4\pi\epsilon_0 R^3} \\ \vec{\nabla} \cdot \vec{E} &= \frac{3q}{4\pi\epsilon_0 R^3} \end{aligned}$$

Recall that $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, so

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi R^3 \rho}{4\pi\epsilon_0 R^3} = \rho/\epsilon_0 \quad \text{inside the sphere! } \checkmark$$

Check outside the sphere: $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{q}{4\pi\epsilon_0 r^2} \right) + 0 + 0 = 0 \quad \text{outside the sphere! } \checkmark$$

(since $r > R$ and we never get to $r = 0$), as expected.