

Lecture #7 Summary

PHYS 313 Electricity and Magnetism I

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1 The Curl of $\vec{E}$ by Direct Differentiation

We know $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, so now:

What is the curl, or vorticity, of an electrostatic field?

Start with

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}')}{|\vec{z}|^2} \hat{\mathcal{R}} d\tau'$$

Recall that $\vec{z} = \vec{r} - \vec{r}'$.

Take the curl of both sides with respect to $\vec{r}$:

$$\vec{\nabla} \times \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{r}') \vec{\nabla} \times \left(\frac{\hat{\mathcal{R}}}{|\vec{z}|^2} \right) d\tau'$$

What is $\vec{\nabla} \times \left(\frac{\hat{\mathcal{R}}}{|\vec{z}|^2} \right)$? ($\vec{z} = \vec{r} - \vec{r}'$)

This is true for $\vec{r}' = 0$, for example, so calculate:

$$\begin{aligned} \vec{\nabla} \times \left(\frac{\hat{r}}{r^2} \right) &= 0 \hat{r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{1}{r^2} \right) \hat{\theta} - \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r^2} \right) \hat{\phi} \\ &= 0 \end{aligned}$$

One worries about $r = 0$ — maybe there is curl at $r = 0$!

Now, just to make sure there is not a delta function lurking at the origin, use a result from Problem 1.61(b), a corollary of the fundamental theorems of calculus (divergence theorem):

$$\int_V (\vec{\nabla} \times \vec{v}) d\tau = - \oint_S \vec{v} \times d\vec{a}$$

(Note that those are integrals of vectors!)

If the curl is truly zero in a volume V , then the integral of $\vec{\nabla} \times d\vec{a}$ over the surface should also be zero.

$$\int_V \vec{\nabla} \times \left(\frac{\hat{r}}{r^2} \right) d\tau = 0 \stackrel{?}{=} - \oint_S \left(\frac{\hat{r}}{r^2} \right) \times d\vec{a}$$

Integrate over the surface of a sphere of radius R . In this case

$$d\vec{a} = R^2 \sin \theta d\theta d\phi \hat{r}$$



Note that the cross product is zero everywhere on the surface of the sphere since the two vectors are parallel:

$$\left(\frac{\hat{r}}{R^2}\right) \times (R^2 \sin \theta d\theta d\phi \hat{r}) = 0$$

Hence

$$\oint_S \left(\frac{\hat{r}}{r^2}\right) \times d\vec{a} = 0$$

so there is no delta function lurking here, and the original conclusion is true:

$$\vec{\nabla} \times \left(\frac{\hat{r}}{r^2}\right) = 0$$

Thus we conclude that $\boxed{\vec{\nabla} \times \vec{E} = 0}$ for an electrostatic field. Later, we will find that the curl is non-zero when there is a time-dependent magnetic field at that point, but for now let's keep it simple and electro-static!

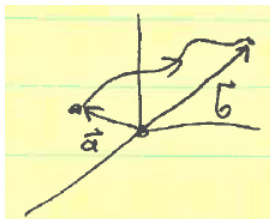
2 The Curl of $\vec{E}$ by Stokes' Theorem

To double check the above result, let's see if we can derive it by a different method. This is "gold standard science"! Consider a point charge at the origin and its electric field

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}$$

Calculate the line integral of $\vec{E}$ from arbitrary locations $\vec{a}$ to $\vec{b}$:

$$\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell}$$



The differential element of length in spherical coordinates is (inside front cover):

$$d\vec{\ell} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$$

so

$$\vec{E} \cdot d\vec{\ell} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot (dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}) = \frac{q}{4\pi\epsilon_0} \frac{dr}{r^2}$$

so

$$\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} = \frac{q}{4\pi\epsilon_0} \int_{r_a}^{r_b} \frac{dr}{r^2} = -\frac{q}{4\pi\epsilon_0} \frac{1}{r} \Big|_{r_a}^{r_b} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right)$$

where r_a, r_b are the radial distances of points $\vec{a}$ and $\vec{b}$ from the origin.

For a closed-loop path, $r_a = r_b$ or $\vec{a} \rightarrow \vec{b}$, the answer is zero:

$$\oint \vec{E} \cdot d\vec{\ell} = 0$$

If this is the case, then Stokes' theorem says

$$\oint_L \vec{E} \cdot d\vec{\ell} = 0 = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$$

where S is any surface with perimeter curve L . Since this is true for any surface S , it must be that the integrand is zero:

$$\vec{\nabla} \times \vec{E} = 0$$

This proof is correct for a single point charge. If there are many point charges, the principle of superposition holds, and

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N$$

so

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{E}_1 + \vec{\nabla} \times \vec{E}_2 + \cdots + \vec{\nabla} \times \vec{E}_N$$

and each is separately zero, so

$$\boxed{\vec{\nabla} \times \vec{E} = 0 \quad \text{for any static charge distribution whatsoever.}}$$

3 Electric Potential

As we saw in the review of mathematics, a curl-free vector field can be written as the gradient of a scalar function (plus an arbitrary constant c):

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}(V + c)$$

since

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla}V) + \vec{\nabla} \times \vec{\nabla}c = 0$$

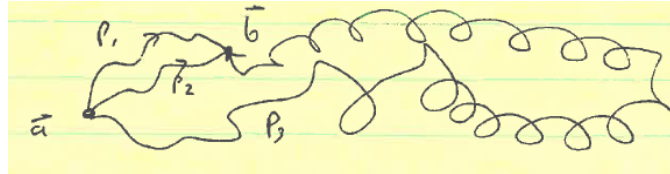
Thus the problem of finding the vector function $\vec{E}$ can be reduced to the problem of finding a scalar function of position V for a given charge distribution. This can be a MUCH EASIER problem than finding $\vec{E}$ directly. Once V is found, we can recover $\vec{E}$ as

$$\vec{E} = -\vec{\nabla}V.$$

How do we define this function V ?

Because $\vec{\nabla} \times \vec{E} = 0$ for static charge distributions, the line integral of $\vec{E}$ is independent of path:

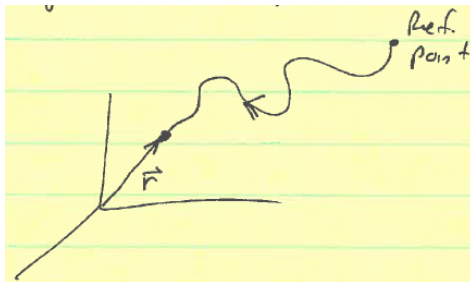
$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \oint_{\text{path}} \vec{E} \cdot d\vec{\ell} = 0 \Rightarrow \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} \text{ is independent of path between the two points } \vec{a} \text{ and } \vec{b}.$$



With this, we can define a unique scalar potential V as

$$V(\vec{r}) \equiv - \int_{\text{Ref. point}}^{\vec{r}} \vec{E} \cdot d\vec{\ell}$$

(Note the minus sign.)

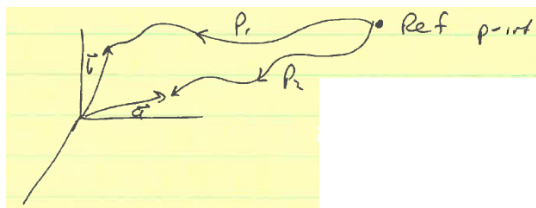


The absolute value of the potential is not physically significant since a constant can be added to $V(\vec{r})$ and $\vec{E}$ does not change.

It is only differences, or gradients, in potential that have physical significance.

(1) Differences in $V(\vec{r})$

Potential difference between locations $\vec{a}$ and $\vec{b}$:



$$V(\vec{a}) = - \int_{\text{Ref. pt.}}^{\vec{a}} \vec{E} \cdot d\vec{\ell}$$

$$V(\vec{b}) = - \int_{\text{Ref. pt.}}^{\vec{b}} \vec{E} \cdot d\vec{\ell}$$

$$V(\vec{b}) - V(\vec{a}) = - \int_{\text{Ref. pt.}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} + \int_{\text{Ref. pt.}}^{\vec{a}} \vec{E} \cdot d\vec{\ell}$$

$$V(\vec{b}) - V(\vec{a}) = - \int_{\text{Ref. pt.}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} - \int_{\vec{a}}^{\text{Ref. pt.}} \vec{E} \cdot d\vec{\ell}$$

$V(\vec{b}) - V(\vec{a}) = - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} \quad \text{Independent of the reference point!}$

(2) Gradients

The fundamental theorem for gradients states

$$V(\vec{b}) - V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} (\vec{\nabla} V) \cdot d\vec{\ell}$$

So combining with the above, we get

$$\int_{\vec{a}}^{\vec{b}} (\vec{\nabla} V) \cdot d\vec{\ell} = - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell}$$

Since this is true for any points $\vec{a}$ and $\vec{b}$, the integrands must be equal:

$\vec{E} = -\vec{\nabla} V$

The potential also obeys the superposition principle. The total potential is the sum of the potentials due to each individual charge:

$$V = V_1 + V_2 + \dots + V_N$$

The SI units of electric field:

$$[E] = \frac{[F]}{[Q]} = \frac{\text{Newtons}}{\text{Coulomb}}$$

The SI units of electric potential:

$$[V] = [E][L] = \frac{\text{Newton-meter}}{\text{Coulomb}} = \frac{\text{Joule}}{\text{Coulomb}}$$