

Lecture #5 Summary

PHYS 313 Electricity and Magnetism I

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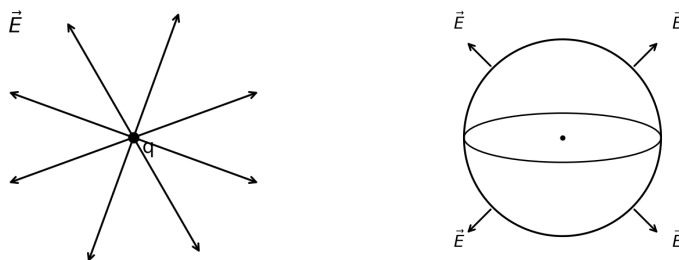
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1 Statement of Gauss's Law

Gauss's Law relates the flux of electric field through a closed surface to the total charge enclosed:

$$\oint_{\text{surface}} \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \quad (1)$$

This can be easily verified (not proven) for a point charge at the origin and a spherical surface centered on the charge.

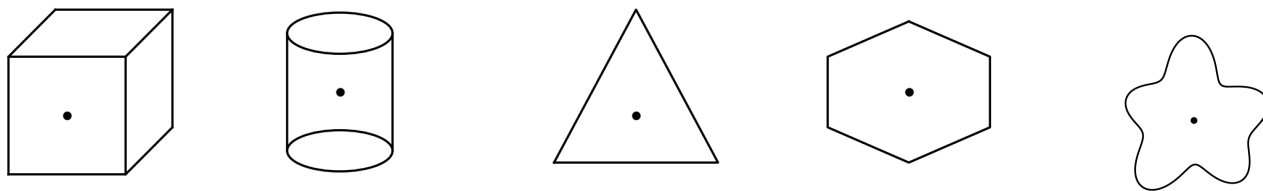


$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r} \quad (2)$$

The LHS of Gauss's law (the flux) through a sphere of radius R is

$$\begin{aligned} \Phi_E &= \oint_{\text{sphere}} \vec{E} \cdot d\vec{a} \\ &= \int_0^{2\pi} \int_0^\pi \frac{q}{4\pi\epsilon_0 R^2} \hat{r} \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}) \\ &= \frac{q}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi \sin\theta \, d\theta \, d\phi \\ &= \frac{q}{4\pi\epsilon_0} (4\pi) = \frac{q}{\epsilon_0} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \end{aligned}$$

(The area element on the sphere of radius R in spherical coordinates is $d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$, and the double integral of $\sin\theta \, d\theta \, d\phi$ is the solid angle subtended by all directions.) Notice that the R^2 cancels. This is true for a spherical surface of any radius, or for any closed surface which encloses the charge. This includes the surfaces shown below.



Gauss's Law is really a re-statement of Coulomb's Law. It depends critically on the fact that the Coulomb force is an inverse square law, and that the area of a sphere scales as radius squared. This cancellation of the r -dependence in the flux integral is what makes Gauss's Law so useful.

Suppose the charge is outside the surface. You get equal amounts of positive and negative flux contributions from the entry and exit paths of the electric field lines (remember that the r^2 part of the Φ_E calculation cancels out).

When adding up all of these positive and negative contributions, you end up with zero:

$$\Phi_E = 0 \quad \text{if no charge is enclosed.}$$

If many charges are present, the above arguments work by linear superposition of the electric fields.

2 Applications of Gauss's Law

Although Gauss's law is always true, it is not always useful! It works well in only 3 situations of high symmetry:

1. Spherical Symmetry
2. Cylindrical Symmetry
3. Plane Symmetry

3 Example use of Gauss's law: Problem 2.16

A hollow spherical shell carries charge density $\rho = k/r^2$ in the region $a \leq r \leq b$, independent of θ, ϕ .

Find the electric fields in the three regions (i) $r < a$, (ii) $a < r < b$, and (iii) $r > b$. Plot $|\vec{E}|$ as a function of r .

The charge density is $\rho(r = a) = k/a^2$, $\rho(r = b) = k/b^2$.

(i) $r < a$

Place a spherical Gaussian surface centered on the origin with radius $R < a$. This surface encloses no charge, so

$$\oint \vec{E} \cdot d\vec{a} = 0 \quad \Rightarrow \quad \vec{E} = 0 \quad \text{for } r < a \quad (3)$$

(We use our *intuition* to note that because the charge distribution is spherically symmetric, the electric field $\vec{E}$ must also be spherically symmetric, $\vec{E} \sim E_0 \hat{r}$, and $\oint \vec{E} \cdot d\vec{a} = E_0 \cdot 4\pi R^2 = 0$.)

(ii) $a < r < b$

Use a Gaussian surface centered on the origin. Now evaluate

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0}, \quad a < R < b \quad (4)$$

The electric field is radial $\vec{E}(R) \sim E(R)\hat{r}$, since the charge distribution is spherically symmetric.

$$\begin{aligned} \Phi_E &= \oint \vec{E}(R) \cdot d\vec{a} = E(R) \int_0^{2\pi} \int_0^\pi \hat{r} \cdot (R^2 \sin \theta d\theta d\phi \hat{r}) \\ \Phi_E &= E(R) R^2 4\pi \end{aligned}$$

The enclosed charge depends on the radius R :

$$Q_{\text{enc}} = \int \rho d\tau = \int_a^R \int_0^{2\pi} \int_0^\pi \frac{k}{r^2} r^2 dr \sin \theta d\theta d\phi$$

$$\begin{aligned} Q_{\text{enc}} &= k r \Big|_a^R 4\pi \\ &= 4\pi k(R - a) \quad a < R < b \end{aligned}$$

Now Gauss's Law gives:

$$\begin{aligned} \Phi_E &= \frac{Q_{\text{enc}}}{\epsilon_0} \\ E(R) 4\pi R^2 &= \frac{4\pi k}{\epsilon_0} (R - a) \end{aligned}$$

So

$$E(R) = \frac{k}{\epsilon_0} \frac{R - a}{R^2}, \quad a < R < b$$

and

$$\vec{E}(R) = \frac{k}{\epsilon_0} \frac{R - a}{R^2} \hat{r}$$

(iii) $r > b$

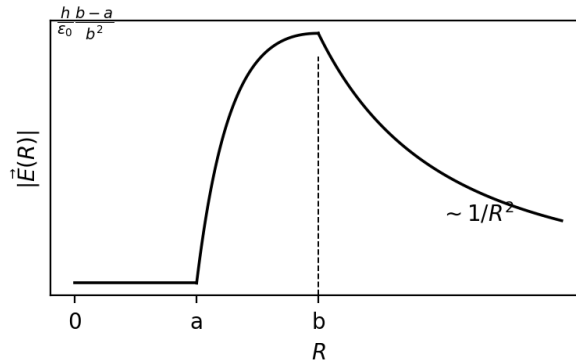
This is the same as above, except that the enclosed charge is that for $R = b$:

$$Q_{\text{enc}} = 4\pi k(b - a)$$

Now Gauss's Law gives:

$$\begin{aligned} E(R) R^2 4\pi &= \frac{4\pi k(b - a)}{\epsilon_0} \\ E(R) &= \frac{k}{\epsilon_0} \frac{(b - a)}{R^2} \\ \vec{E}(R) &= \frac{k}{\epsilon_0} \frac{(b - a)}{R^2} \hat{r} \end{aligned}$$

The electric field looks like this:



Experimental tests of the validity of the inverse square law depend on the absence of a force on a charge located inside a charged shell. This absence of a force is independent of the distribution of charge on the sphere.

There have been experimental tests of deviations of Coulomb's law from the $1/r^2$ dependence.

There are theoretical predictions that gravity leaks into extra dimensions, so its form may be modified such that it becomes very weak in 4 spacetime dimensions, perhaps leading to deviations from the $1/r^2$ distance dependence of the gravitational force. Inverse square laws are also discussed in detail on Wikipedia

4 Differential Form of Gauss's Law

We can get a differential form of Gauss's Law by applying the divergence theorem:

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} d\tau \quad (5)$$

We can write the enclosed charge in terms of a volume integral

$$Q_{\text{enc}} = \int_V \rho d\tau \quad \text{where } \rho \text{ is the charge density in the volume } V. \quad (6)$$

So Gauss's Law becomes

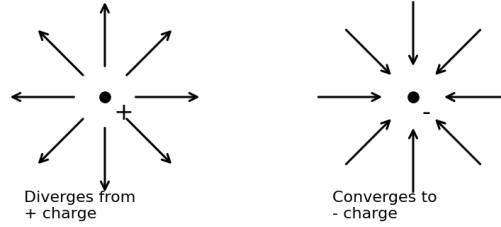
$$\Phi_E = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$\int_V \vec{\nabla} \cdot \vec{E} d\tau = \frac{1}{\epsilon_0} \int_V \rho d\tau$$

Both of these integrals are over the same volume V , and this volume can be chosen arbitrarily. Hence the integrands must be equal:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (7)$$

This is the differential form of Gauss's Law.



Electric field diverges from a positive charge, and converges on a negative charge. The electric field lines show the direction of the electric field. Their density shows the magnitude of the electric field.

5 Checking Gauss's Law for a Point Charge

Try Gauss's Law on the point charge at the origin:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \quad (8)$$

$$\vec{\nabla} \cdot \vec{E} = \frac{q}{4\pi\epsilon_0} \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) \quad (9)$$

But we have seen before that

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi\delta^3(\vec{r}) \quad (10)$$

So

$$\vec{\nabla} \cdot \vec{E} = \frac{q}{4\pi\epsilon_0} 4\pi\delta^3(\vec{r}) = \frac{q}{\epsilon_0} \delta^3(\vec{r}) = \frac{\rho}{\epsilon_0} \quad (11)$$

Hence the expression for the charge density of a point charge must be:

$$\rho = q\delta^3(\vec{r}) \quad \text{point charge at the origin.} \quad (12)$$

Note: $\delta^3(\vec{r})$ has the units of inverse volume! Because

$$\int_{\text{all space}} \delta^3(\vec{r}) d\tau = 1 \quad (\text{dimensionless}) \quad (13)$$

Here we use the bracket notation [...] to indicate the units/dimensions of a quantity:

$$[\delta(x)] \sim \frac{1}{L}, \quad [\delta^3(\vec{r})] \sim \frac{1}{L^3}, \quad \text{so } [\rho] \sim \frac{\text{Charge}}{\text{Volume}} \quad (14)$$

where L is the dimension of length.

6 How do we recover charge distribution from the electric field that it produces? Problem 2.9

Suppose the electric field in some region is found to be

$$\vec{E} = k r^3 \hat{r}$$

Find the charge density ρ that gives rise to this electric field.

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 k r^3) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \cdot 0) + \frac{1}{r \sin \theta} \frac{\partial 0}{\partial \phi} \\ &= \frac{k}{r^2} 5r^4\end{aligned}$$

$$\vec{\nabla} \cdot \vec{E} = 5kr^2$$

Hence

$$\frac{\rho}{\epsilon_0} = 5kr^2$$

or

$$\rho(r) = 5\epsilon_0 k r^2 \quad \text{is the charge density}$$

Note that this electric field cannot continue on forever – it would diverge. Something has to change as r increases.