Lecture #4 Summary
PHYS 313 Electricity and Magnetism I

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Electric Charge

1. Electric charge comes in two flavors, arbitrarily called "positive" and "negative," although they could be called "chocolate" and "vanilla."

   The "plus" and "minus" names are useful because these two flavors tend to cancel each other out.

   In fact, bulk objects are very close to being charge neutral, or having zero net charge.

   Gravitational charge only comes in one flavor, as far as we know, so the force of gravity is always attractive. Note that in principle Magnetic charge has two flavors – "North" and "South." However, we can't split these charges apart, they always appear together in a unit called a magnetic dipole. This fact is expressed in the Maxwell equation: $\vec{\nabla} \cdot \vec{B} = 0$. The right hand side says there are no scalar monopole sources of magnetic field.

2. Charge is conserved. It is neither created nor destroyed in any of the classical (non-QED) non-relativistic processes we consider in this course. As far as we are concerned here, charge is conserved both globally and locally in all processes.

   Local conservation of charge $\Rightarrow$ Continuity Equation:

   $$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

   where ρ is charge density, $\vec{J}$ is current density, and t is time. This is actually incorporated in Maxwell's Equations.

3. Charge is quantized. This is an experimental fact. The smallest non-zero charge observed on any "free" particle is the electronic charge

   $$e = 1.602 \times 10^{-19} \text{ Coulombs}$$

   All charges are integer multiples of this charge unit:

   $$Q = \pm Ne$$

   where N is a positive integer or zero. Quarks (which are bound particles inside the nucleus) have charge $\pm\frac{1}{3}e$ or $\pm\frac{2}{3}e$. There are also excitations in condensed matter systems that can have fractional charges (fractional quantum Hall effect), but these objects do not exist as "free" particles.

Coulomb's Law

The force between two charges is proportional to the product of their charges, divided by their separation squared.

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\mathfrak{r}|^2} \hat{\mathfrak{r}}$$

The force acts along the direction separating the two (static) charges, $\hat{\mathfrak{r}}$. Take this as an experimental fact. Here ε_0 is the permittivity of free space:

$$\varepsilon_0 = 8.85 \times 10^{-12} \; \frac{C^2}{Nm^2} = 8.85 \; \frac{pF}{m}$$

$\vec{F}_{12}$ is attractive if q_1 and q_2 have opposite signs.
$\vec{F}_{12}$ is repulsive if q_1 and q_2 have the same sign.
$\vec{F}_{12}$ is zero if q_1 or q_2 is zero.

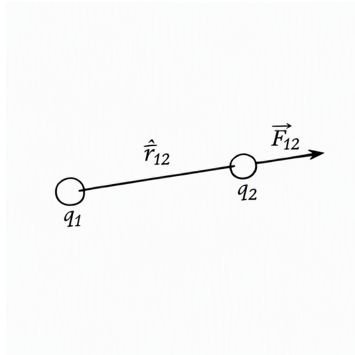

Figure 1: q_1 at the source point, q_2 at the field point, separated by $\vec{r}_{12}$.

Superposition of Forces

What is the force on a test charge due to a collection of charges $q_1, q_2, \ldots, q_N$? (At rest!)

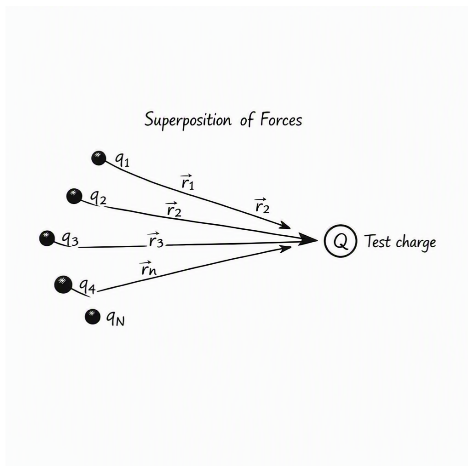

Figure 2: Charges $q_1, \ldots, q_N$ each exert a force on the test charge Q.

The fact that Coulomb's law is linear in the source charge q_i means that a principle of superposition holds for the total force on Q.

The force that q_i exerts on Q is independent of all the other charges present in the problem. (Would not hold, for example, if $F \sim (q_i + Q)^2$.)

Hence

$$F_Q = F_{1Q} + F_{2Q} + F_{3Q} + \cdots + F_{NQ}$$
$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{|\mathfrak{r}_1|^2}\hat{\mathfrak{r}}_1 + \frac{q_2 Q}{|\mathfrak{r}_2|^2}\hat{\mathfrak{r}}_2 + \cdots + \frac{q_N Q}{|\mathfrak{r}_N|^2}\hat{\mathfrak{r}}_N\right)$$
$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1\hat{\mathfrak{r}}_1}{|\mathfrak{r}_1|^2} + \cdots + \frac{q_N\hat{\mathfrak{r}}_N}{|\mathfrak{r}_N|^2}\right)$$

The quantity in parentheses has nothing to do with the test charge, other than its location. It is the "Electric Odor" of the N charges evaluated at the site of the test charge.

The Electric Field

We define this "electric odor" or "electric field" as the force per unit charge at the location of that charge:

$$\vec{E}(\vec{r}) \equiv \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i \hat{\mathfrak{r}}_i}{|\mathfrak{r}_i|^2}$$

where $\mathfrak{r}_i$ is the vector pointing from charge q_i to the field point at position r.
Note that the electric field is defined as the force per unit charge in the limit of the test charge going to zero:

$$\vec{E}(\vec{r}) = \lim_{Q \to 0} \frac{\vec{F}_Q}{Q}$$

Continuous Charge Distributions

If there is a continuous distribution of charge, the sum becomes an integral:

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int dq \, \frac{\hat{\mathfrak{r}}}{|\mathfrak{r}|^2}$$

The continuous charge is usually found in one of three distributions:

1. Line Charge
2. Surface Charge
3. Volume Charge

1. Line Charge

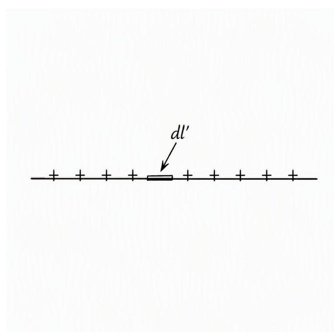

Figure 3: A line charge with element dl'.

Define $\lambda = \dfrac{\text{charge}}{\text{unit length}}$, $\quad dq = \lambda \, dl'$. $\quad$ (Example: $\lambda = 5 \ \mu C/m$)
Then the element of charge dq is the charge per unit length times an element of length dl':

$$dq = \lambda \, dl'$$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{\text{Line}} \frac{\lambda(\vec{r'})}{|\mathfrak{r}|^2} \hat{\mathfrak{r}} \cdot d\vec{l'}$$

Note that λ could be a function of position: $\lambda = \lambda(x')$

2. Surface Charge

Define $\sigma = \dfrac{\text{charge}}{\text{unit area}}$. (Example: $\sigma = 17\ nC/m^2$)

The element of charge dq is the charge per unit area times an element of area da':

$$dq = \sigma\, da'$$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{\text{Surface}} \frac{\sigma(\vec{r'})}{|\mathfrak{r}|^2} \hat{\mathfrak{r}} \cdot d\vec{a'}$$

Note that σ could be a function of position $\sigma(x, y)$

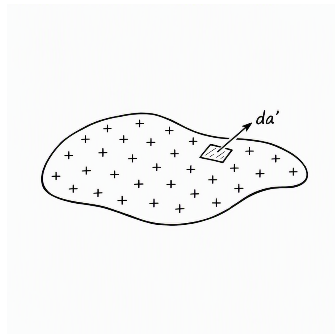

Figure 4: A surface charge with element da'.

3. Volume Charge

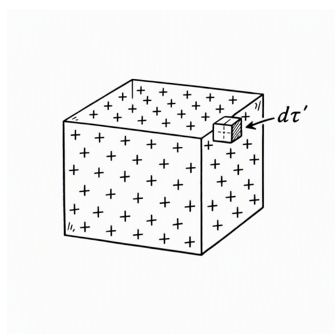

Figure 5: A volume charge with element $d\tau'$.

Define $\rho = \dfrac{\text{charge}}{\text{unit volume}}$. (Example: $\rho = -3.9\ \mu C/m^3$)

The element of charge dq is the charge density ρ times an element of volume $d\tau'$:

$$dq = \rho\, d\tau' \qquad \vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{\text{Volume}} \frac{\rho(r')}{|\mathfrak{r}|^2} \hat{\mathfrak{r}}\, d\tau'$$

ρ could be a function of position $\rho = \rho(x, y, z)$

Does the Field Contain Information about the Source?

Question: Is there any information about the source charges in the electric field $E(r)$?

No and Yes

No: There is no explicit information about the source charges at position r'. All of that information has been integrated out.

Yes: The $\vec{E}(r)$ field has no explicit dependence on the charge distribution, but $\vec{\nabla}\cdot\vec{E}$ does describe the source location through

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$$

This is an amazing property of electric fields! Their divergence tells you the locations of all the charges that gave rise to the field!

Problem 2.6

Find the electric field a distance z above the center of a flat circular disk of radius R, which carries a uniform surface charge σ.

We use the electric field formula for 2D charge distributions:

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int_{\text{Surface}} \frac{\sigma(r')}{|\mathfrak{r}|^2}\hat{\mathfrak{r}}\, da'$$

Here we have $\sigma(r') = \sigma$, a constant.

The surface is a disk of radius R.

From symmetry we know that the direction of $\vec{E}$ is $\hat{z}$ only. There will be equal and opposite contributions to the horizontal component of $\vec{E}$ from opposite sides of the disk.

The net vector field at the field point is up (z-direction). Hence we only need to calculate the vertical component of $\vec{E}$ there.

Also from symmetry, we want to use cylindrical coordinates in the integral. Put the origin in a convenient location! Note that the physics is independent of the choice of location for origin, but some choices are better than others in terms of calculations.

$$E\cdot\hat{z} = E_z = \frac{1}{4\pi\varepsilon_0}\int_{\text{Surface}} \frac{\sigma}{|\mathfrak{r}|^2}\hat{\mathfrak{r}}\cdot\hat{z}\, da'$$

What is $|\mathfrak{r}|^2$? From the triangle between the origin, the source point and field point, we have $|\mathfrak{r}|^2 = s^2 + z^2$

What is $\hat{\mathfrak{r}}\cdot\hat{z}$? It is $\cos\theta$, where $\cos\theta = \dfrac{z}{\sqrt{s^2+z^2}}$ from the geometry of the triangle and the vector $\mathfrak{r}$ between the source point and the field point.

Thus the integral becomes:

$$E_z = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \sigma\frac{z}{\sqrt{s^2+z^2}}\frac{1}{s^2+z^2}\, s\, ds\, d\phi$$

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{z\, s}{(s^2+z^2)^{3/2}}\, ds\, d\phi$$

The ϕ integral is easy, it is just 2π.

$$E_z = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{z\, s}{(s^2+z^2)^{3/2}}\, ds$$

You can look this integral up (it's easy to do!!) Or consult Murray R. Spiegel's *Mathematical Handbook of Formulas and Tables* (Schaum's Outline Series), page 67, integral 14.197:

$$\int \frac{x\, dx}{(x^2+a^2)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}} + C$$

So we have

$$E_z = \frac{\sigma z}{2\varepsilon_0}\left[\frac{-1}{\sqrt{s^2+z^2}}\right]_0^R = \frac{\sigma z}{2\varepsilon_0}\left\{\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right\}$$

The final result is

$$\vec{E}(z) = \frac{\sigma z}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right)\hat{z} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}$$

Now consider several limiting cases and see if they agree with our intuition.

Limit: $R \to \infty$

The disk gets very large, it becomes an infinite plane. In this case

$$\vec{E}(z) = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2 + z^2}}\right]\hat{z} \to \frac{\sigma}{2\varepsilon_0}\hat{z}$$

This is the well known result for the electric field above a charged plane, found from Gauss's Law.

Limit: $z \to \infty$ with R finite

$$\vec{E}(z) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right)\hat{z}$$

Using the binomial expansion $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

$$\frac{z}{z\sqrt{1 + R^2/z^2}} = \frac{1}{\sqrt{1 + R^2/z^2}} = \left(1 + \frac{R^2}{z^2}\right)^{-1/2} \approx 1 - \frac{R^2}{2z^2}$$

where R/z is the small parameter.

$$E(z) \approx \frac{\sigma}{2\varepsilon_0}\left(1 - 1 + \frac{R^2}{2z^2}\right)\hat{z} = \frac{\sigma\pi R^2}{4\pi\varepsilon_0 z^2}\hat{z} = \frac{Q}{4\pi\varepsilon_0 z^2}\hat{z} \qquad (z \gg R)$$

where $Q = \sigma\pi R^2$ is the total charge stored on the disk.

This is just the electric field from a point charge Q a distance $z \gg R$ away.