Lecture #30 Summary
PHYS 313 Electricity and Magnetism I

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What about the curl of $\vec{B}$? Carry it out ...

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left[\vec{J}(\vec{r}\,') \times \frac{\hat{\imath}}{\imath^2} \right] d\tau'$$

Now use the product rule (#8 from the inside front cover of Griffiths):

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

$$\vec{A} = \vec{J}(\vec{r}\,') \qquad \vec{B} = \frac{\hat{\imath}}{\imath^2}$$

$$\vec{\nabla} \times \left(\vec{J}(\vec{r}\,') \times \frac{\hat{\imath}}{\imath^2} \right) = \underbrace{\left(\frac{\hat{\imath}}{\imath^2} \cdot \vec{\nabla} \right) \vec{J}(\vec{r}\,')}_{=0} - \left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla} \right) \frac{\hat{\imath}}{\imath^2} + \vec{J}(\vec{r}\,') \left(\vec{\nabla} \cdot \frac{\hat{\imath}}{\imath^2} \right) - \underbrace{\frac{\hat{\imath}}{\imath^2} \left(\vec{\nabla} \cdot \vec{J}(\vec{r}\,') \right)}_{=0}$$

Two terms are zero because $\vec{\nabla}$ acts on $\vec{r}$, not $\vec{r}\,'$ and $\vec{J}$ is only a function of $\vec{r}\,'$.
So we have:

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}\,') \left(\vec{\nabla} \cdot \frac{\hat{\imath}}{\imath^2} \right) d\tau' - \frac{\mu_0}{4\pi} \int \left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla} \right) \frac{\hat{\imath}}{\imath^2} \, d\tau'$$

The divergence is our old friend from Chapter 1:

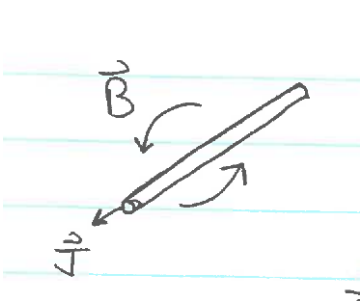
$$\boxed{\vec{\nabla} \cdot \left(\frac{\hat{\imath}}{\imath^2} \right) = 4\pi \, \delta^3(\vec{\imath})}$$

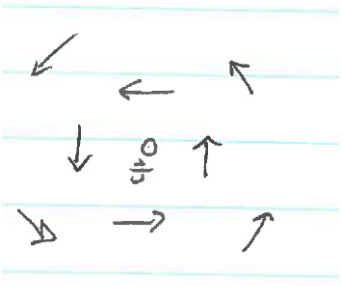
The second term integrates to zero (see page 230 of Griffiths). Thus we get

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}\,') \, 4\pi \, \delta^3(\vec{\imath}) \, d\tau' \\ &= \mu_0 \int \vec{J}(\vec{r}\,') \, \delta^3(\vec{r} - \vec{r}\,') \, d\tau' \end{aligned}$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \, \vec{J}(\vec{r})}$$

This says that currents $\vec{J}$ are the (vector) source of magnetic field. A current will give rise to a curl in $\vec{B}$.





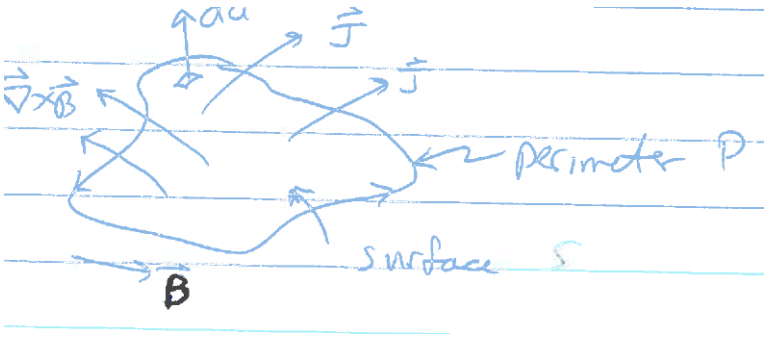
Ampere's Law in Integral Form

We have found: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

In other words, the magnetic field curls around a current.

We can convert this to an integral form by integrating both sides over an arbitrary surface S and using Stoke's theorem:

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$



We integrate both vectors dotted into a differential area element $d\vec{a}$ over the open surface S. Note that the surface has a closed perimeter P.

Now use Stoke's Theorem to convert the surface integral to a perimeter integral:

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \oint_P \vec{B} \cdot d\vec{l}$$

Meanwhile, the RHS of Ampere's Law becomes the total current piercing this loop, through the surface S:

$$\mu_0 \int_S \vec{J} \cdot d\vec{a} = \mu_0 \, I_{enc}$$

where I_{enc} is the total current going through the surface bounded by the perimeter loop. Recall that this integral gives the same result independent of the shape of S, as long as each surface terminates on the perimeter loop P.

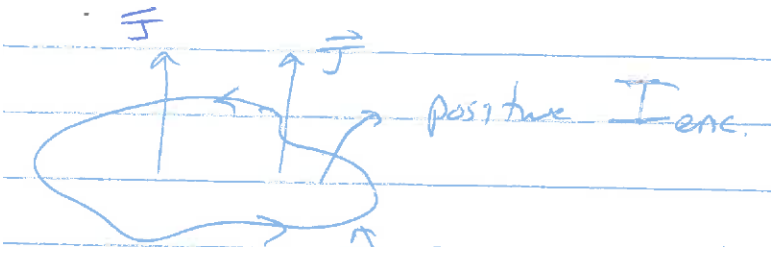
Therefore we have

$$\boxed{\oint_P \vec{B} \cdot d\vec{l} = \mu_0 \, I_{enc}}$$

This integral form of Ampere's Law is very useful when the magnetic field has enough symmetry.

This Law is similar to Gauss's Law in electrostatics.

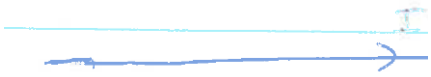
Use the right hand rule to determine the direction of the loop integration and the direction of positive current flow:



The perimeter path P is sometimes called an Amperian Loop (in analogy to a Gaussian surface in electrostatics)

Example use of Ampere's Law in integral form

Find the magnetic field a distance s from a long straight wire carrying steady current I.



We know this problem has cylindrical symmetry, so we shall use cylindrical coordinates (s, ϕ, z).

A Symmetry group is a set of operations that leave the system invariant, meaning that it looks and acts the same as before the operation was performed. We expect that the physical phenomena (such as the magnetic field) will also obey these symmetry properties. Here is a list of symmetries for the long current-carrying wire:

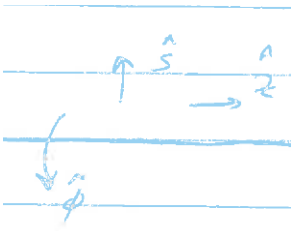
1) Any rotation around the wire axis

2) Mirror plane $\perp$ to the wire (This is an anti-symmetry in the sense that it turns the current I into $-I$, and thus should also change the sign of $\vec{B}$, as we know from the Biot-Savart law)

3) Translational symmetry along the wire

4) Any mirror plane that contains the wire

The possible components of magnetic field $\vec{B}$ are $\hat{s}$, $\hat{\phi}$, $\hat{z}$



Having a $\hat{z}$-component of $\vec{B}$ is not compatible with the translational symmetry and the mirror plane $\perp$ to the wire. The perpendicular mirror plane anti-symmetry means that $B_z = -B_z$, leaving us with $B_z = 0$. Also, from the Biot-Savart law we see that $\vec{B} \perp \vec{I}$ (when $\vec{I} \sim \hat{z}$ is uni-directional, as it is in this case)

What about the $\hat{s}$ component of $\vec{B}$? Consider changing the direction of $\vec{I}$. This will reverse the sign of $\vec{B}$ (from Biot-Savart, Eq. 5.34). In particular the $\hat{s}$ component of $\vec{B}$ changes sign. But changing the current direction is equivalent to a mirror operation through the plane $\perp$ to the wire. This operation leaves the $\hat{s}$ component of $\vec{B}$ unchanged. Hence this would lead to a contradiction for a non-zero s-component, meaning that $B_s = 0$.

This leaves only the ϕ component, $\vec{B} = B(s, \phi, z)\hat{\phi}$.

What coordinates does this component depend on?

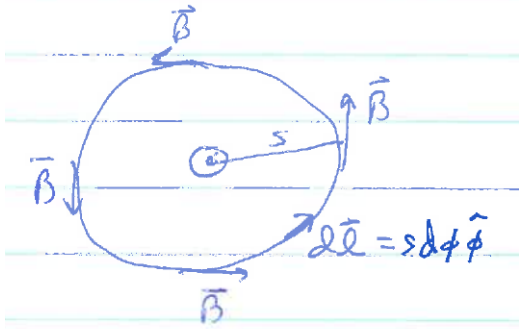
Because of the translational symmetry of the wire, $\vec{B}$ can have no z-dependence since every location along the length of the wire is indistinguishable from any other point.

Because of the azimuthal symmetry of the problem (i.e. it is invariant for any rotation about the z-axis), $\vec{B}$ can have no ϕ-dependence

Hence we are left with,

$$\boxed{\vec{B} = B(s)\,\hat{\phi}}$$

The field must be purely circumferential (i.e. in the $\hat{\phi}$ direction) It is constant for a fixed distance s from the wire. So make the Amperian loop a circle centered on the wire:



Perform the integral in Ampere's law: $\oint_P \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$. The Amperian loop is a circle of radius s and is parameterized by the angle ϕ running from 0 to 2π.

$$\vec{B} = B(s)\,\hat{\phi}$$
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$
$$\int_0^{2\pi} B(s)\,\hat{\phi} \cdot \left(s\,d\phi\,\hat{\phi}\right) = \mu_0 I$$
$$B(s)\,s\,2\pi = \mu_0 I$$
$$B(s) = \frac{\mu_0 I}{2\pi s}$$

$$\boxed{\vec{B}(s) = \frac{\mu_0 I\,\hat{\phi}}{2\pi s}}$$

The same result we obtained before, but now with much less calculational effort. However, we need strong physics intuition to set the problem up.

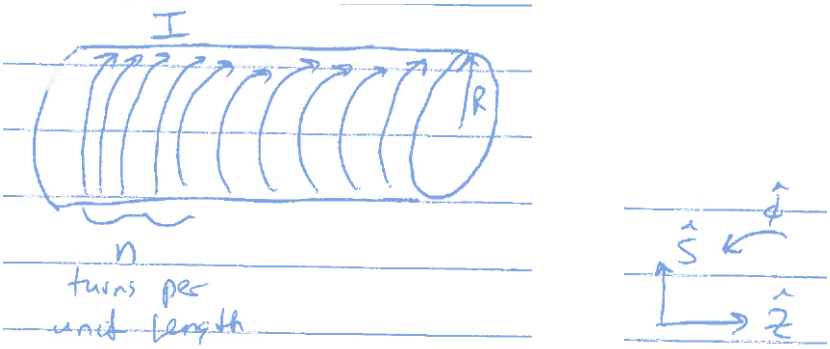
The Infinite Solenoid. Another example of Ampere's Law

Consider a solenoid made up of n turns per unit length of very fine wire. The radius of the solenoid is R. The wire carries a steady current I. What is the magnetic field produced by the solenoid everywhere in space?

The solenoid is a practical means of generating large homogeneous magnetic fields. They are used for MRI, NMR spectroscopy, and for experimental physics. Most magnets that produce more than 1 Tesla of magnetic field are made using superconducting wire. Superconductors have zero resistance for steady (DC) currents. This allows superconducting magnets to operate in persistent current mode. In practice the solenoid is connected to a current source that "charges" it up with current (typically more that 100 Amps), and then a persistent current switch is thrown to allow the current to circulate through the solenoid, disconnecting it from the outside current source. This current will then persist forever, as long as the superconductor remains cold.

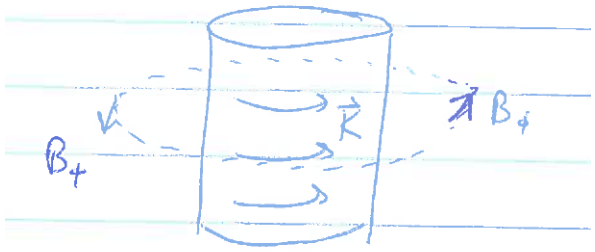
We will also use cylindrical coordinates here. Again we will use symmetry arguments to deduce the form of $\vec{B}(s, \phi, z)$.

First, let's figure out what is the direction of the magnetic field?



From Biot-Savart we know that $\vec{B} \perp \vec{I}$ for uni-directional $\vec{I}$. In this case $\vec{I} \sim \hat{\phi}$, hence $\vec{B}$ can only have $\hat{z}$ and $\hat{s}$ components.

Lets suppose there is a $\hat{\phi}$ component. Create an Amperian loop centered on the solenoid axis, inside or outside the solenoid.



The current in the solenoid can be considered to be a surface current $\vec{K} = n\vec{I} = nI\hat{\phi}$
In this case we expect B_ϕ to be independent of ϕ. Carry out the Amperian loop calculation:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

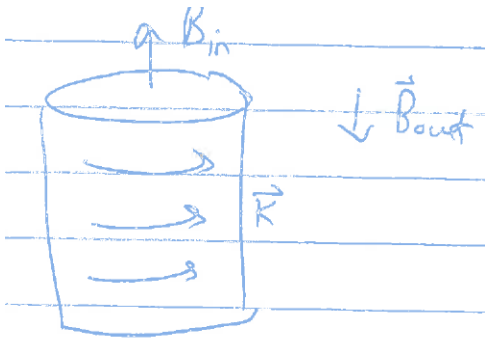
$B_\phi \, 2\pi s = 0$ since no current pierces the surface.

$\Rightarrow B_\phi = 0$

What about an $\hat{s}$ component? For the same reason as the wire, there is no $\hat{s}$ component of $\vec{B}$. If we reverse the direction of the current, the $\hat{s}$ component should also reverse, according to Biot-Savart. But this is equivalent to flipping the magnet over (sending $\hat{z}$ to $-\hat{z}$), which leaves the $\hat{s}$-component invariant. These two observations make contradictory demands on the radial component, leading us to conclude that $\vec{B}$ has no $\hat{s}$ component.

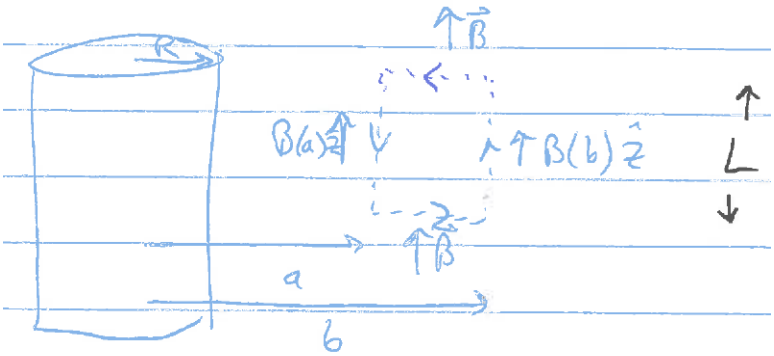
Hence the magnetic field in and around the solenoid must be in the z-direction only. From the right-hand-rule we expect:

$$\vec{B} = B_z(s)\,\hat{z}$$



Find $\vec{B}_{out}$ first, then $\vec{B}_{in}$.

But, Use an Amperian loop entirely outside the solenoid. The loop is a rectangle in the (s, z) plane, running from $s = a$ to $s = b$ and having length L along z:



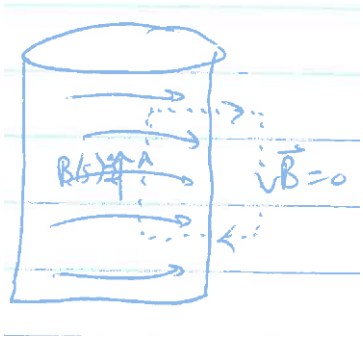
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$0 + 0 + B(b)\, L - B(a)\, L = 0$$

Hence $B(a) = B(b)$. Thus $\vec{B}$ is independent of s outside the solenoid!

We expect $B(s \to \infty) \to 0$. But if we make $b \to \infty$, then $B(b) = B(a) = 0$.
Hence in this ideal situation, there is no magnetic field outside the solenoid!

Now look inside the solenoid to find B_{in}: Put the Amperian loop partly inside and partly outside the solenoid. THere is now current piercing the loop:



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$0 + 0 + 0 + B(s)\, L = \mu_0\, n\, I\, L$$

so

$$B(s) = \mu_0\, n\, I \qquad \text{inside the solenoid, also independent of } s.$$

In Summary:

$$\boxed{\vec{B}(s) = \begin{cases} \mu_0\, n\, I\, \hat{z} & \text{inside} \\ 0 & \text{outside} \end{cases}}$$

The magnetic field is independent of position inside the solenoid as well!

The magnetic field can be made very strong by increasing the current I and the number of turns per unit length n.

Let's estimate the field inside a superconducting magnet. In a typical superconducting magnet:

$$n = 10^4 \text{ turns/m} \qquad \text{Requires thin wire and no heating!}$$
$$I = 10^2 \text{ A} \qquad \Rightarrow \text{ Superconductor}$$

$$B = \mu_0\, n\, I = 1.26\, \frac{\text{N}}{\text{A}^2}\, \frac{1}{\text{m}}\, \text{A} = 1.26\, \frac{\text{N}}{\text{A m}} = 1.26 \text{ T}$$

The Earth's magnetic field is about

$$B_{earth} \sim 50\ \mu\text{T}$$

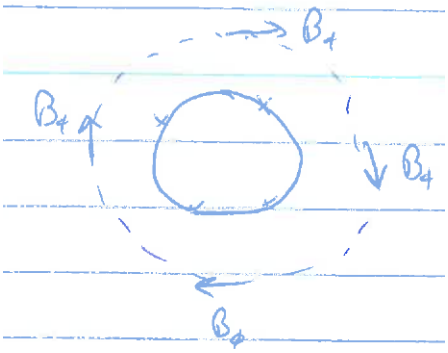
MRI Magnets for human imaging typically have $B \sim 1$–2 T.
The strength of NMR signals scales like B^2, so magnets for NMR spectroscopy are typically at fields of $5 - 20$ T.

Example calculation: Problem 5.14

a) Consider a steady current I flowing on just the surface of a cylindrical wire of radius a. What is $\vec{B}$ inside and outside the wire?



Like the wire problem, we expect only a $\hat{\phi}$ component of $\vec{B}$, and no $\hat{z}$ or $\hat{s}$ components.
Outside. Draw an Amperian loop.



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\int_0^{2\pi} \left(B_\phi(s)\,\hat{\phi}\right) \cdot \left(s\,\hat{\phi}\,d\phi\right) = \mu_0 I$$

$$B_\phi(s)\, s\, 2\pi = \mu_0 I$$

$$B_\phi = \frac{\mu_0 I}{2\pi s}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi},$$

which is the same as having a filamentary wire at the center of the cylinder. This is reminiscent of Gauss's law for the electric field outside a uniformly charged sphere, for example.

Inside the wire, draw a similar Amperian loop. Now it contains no current. However, the B_ϕ integral produces

$$\oint \vec{B} \cdot d\vec{l} = B_\phi(s)\, 2\pi s = 0$$

$$\Rightarrow B_\phi(s) = 0 \qquad \text{inside the wire}$$

To summarize

$$\boxed{\vec{B}(s) = \begin{cases} \dfrac{\mu_0 I}{2\pi s}\,\hat{\phi} & s > a \\ 0 & s < a \end{cases}}$$

b) Now suppose that the current density varies linearly with radius inside the wire, $J \sim s$. What is the proportionality constant?

$$J = c\,s$$

$$\int_S \vec{J} \cdot d\vec{a} = I$$

$$\int_0^{2\pi} \int_0^a c\, s\; s\, ds\, d\phi = I$$

$$c \int_0^{2\pi} d\phi \int_0^a s^2\, ds = I$$

$$c\, 2\pi \frac{1}{3} s^3 \Big|_0^a = I$$

$$c \frac{2\pi a^3}{3} = I$$

$$c = \frac{3I}{2\pi a^3}$$

So we can know write the current density as,

$$\boxed{\vec{J} = \frac{3Is}{2\pi a^3} \hat{z}}$$

What is the field outside? If we draw a circular Amperian loop at $s > a$ we get the same thing as above. The magnetic field will be

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\, \hat{\phi} \qquad s > a$$

For the field inside, we draw the same Amperian loop as above, but the enclosed current is now non-zero:

$$I_{\text{enc}} = \int_0^s J(s')\, da = \int_0^s cs'(2\pi s')\, ds' = \frac{2\pi c s^3}{3} = I \frac{s^3}{a^3}, \text{ for } s < a;$$

$$\text{So } \vec{B} = \begin{cases} \frac{\mu_0 I s^2}{2\pi a^3} \hat{\phi}, & \text{for } s < a; \\ \frac{\mu_0 I}{2\pi s} \hat{\phi}, & \text{for } s > a. \end{cases}$$