Lecture #2 Summary
PHYS 313 Electricity and Magnetism I

Prof. Steven Anlage

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Integral Calculus

There are 3 kinds of integrals that one can do with vector fields:

1. Line integrals
2. Surface integrals
3. Volume integrals

In all 3 cases, you form a scalar quantity and then integrate it over some specified geometrical object (line, surface, or volume).

1. Line Integrals

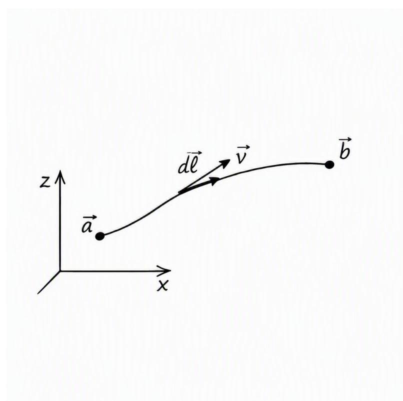

Figure 1: A line integral: at each point along the path, take the dot product of $\vec{v}$ with a differential displacement vector $d\vec{l}$ tangent to the path, then add them all up.

$$\int_{\vec{a}}^{\vec{b}} \vec{v} \cdot d\vec{l}$$

$\vec{v}$ is a vector function of position.
$d\vec{l}$ is an infinitesimal displacement vector (this is a bit over-simplified - it should be directed tangent to the path):

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

The path is a line path connecting points $\vec{a}$ and $\vec{b}$.
A familiar example from mechanics is the calculation of work done by a force applied through a displacement:

$$W = \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{l}$$

We will focus on vector functions whose line integral values are independent of path, and depend only on the end points. A force with this property is said to be conservative.

2. Surface Integrals

$$\int_S \vec{v} \cdot d\vec{a}$$

$\vec{v}$ is a vector function of position.
$d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface S.

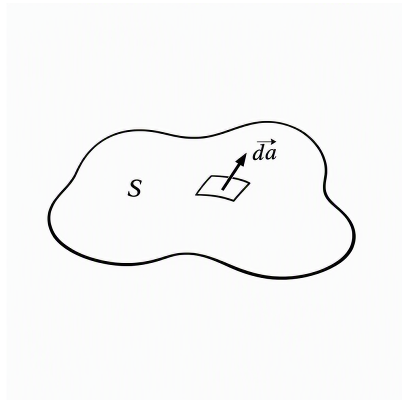

Figure 2: A surface integral: $d\vec{a}$ is perpendicular to the surface S.

This integral is sometimes called the "flux" of $\vec{v}$ through the surface S.

Some functions have a value for the surface integral that is independent of the choice of surface, depending only on the value of the function on the boundary line.

Consider a soap bubble created on a wire ring. The metal ring is a fixed perimeter or circuit that provides a frame for supporting the open surface. The soap bubble (surface) can be distorted into many different shapes by blowing on it. However, this family of surfaces will all have the same perimeter curve.

3. Volume Integrals

$$\int_V Q \, d\tau$$

$Q = Q(x, y, z)$ is a scalar function of position. $d\tau = dx\, dy\, dz$ is a differential volume element in Cartesian coordinates.

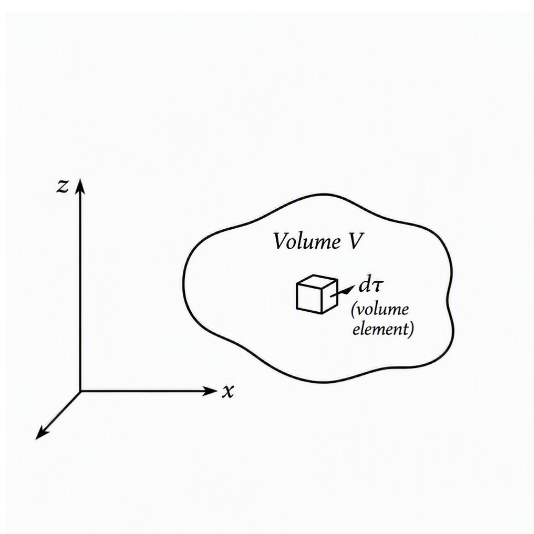

Figure 3: A volume integral: $d\tau$ is a differential element of the volume V.

The Fundamental Theorem of Calculus

For a function $f(x)$ of one variable,

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

The integral of a derivative over an interval is given by the value of the function at the endpoints (boundaries). The integral is basically the anti-derivative.

This fundamental theorem can be generalized to:

1. Gradients
2. Divergences
3. Curls

(See Appendix A for proofs.)

1. Fundamental Theorem for Gradients

$$\int_a^b \vec{\nabla} Q \cdot d\vec{l} = Q(\vec{b}) - Q(\vec{a})$$

for a scalar function $Q(x, y, z)$.

Corollary 1: $\int_{\vec{a}}^{\vec{b}} \vec{\nabla} Q \cdot d\vec{l}$ is independent of the path taken from $\vec{a}$ to $\vec{b}$.

Corollary 2: $\oint \vec{\nabla} Q \cdot d\vec{l} = 0$ for any closed loop. (True because $\vec{\nabla} \times (\vec{\nabla} Q) = 0$ for all Q, by Stokes' theorem.)

2. The Fundamental Theorem for Divergences

For a vector function $\vec{v}(x, y, z)$:

$$\int_V \vec{\nabla} \cdot \vec{v}\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

This is known as Gauss's Theorem, Green's Theorem, or the Divergence Theorem. Note that the surface S is that which encloses and delineates the volume V.

The integral of a derivative (divergence) over a region (volume) is equal to the value of the function at the boundary (surface that bounds the volume).

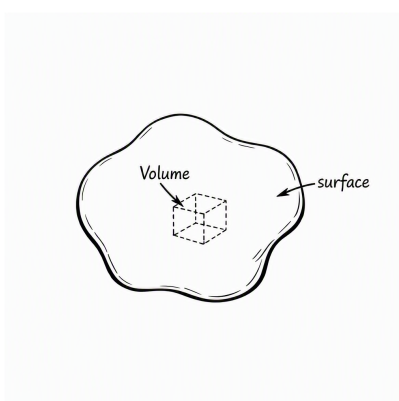

Figure 4: A volume bounded by a surface. This will be used to express Gauss's Law of electrostatics.

If we interpret $\vec{v}$ as the flow velocity of an incompressible fluid, then the flux of $\vec{v}$ through the surface (RHS) is equal to the total amount of fluid passing through the surface per unit time. This is equal to the volume-integrated divergence of the fluid flow, i.e. the number of faucets or sources of fluid in the volume.

There are no corollaries to this theorem because the surface and volume are constrained — not like 2 points defining a line, or a closed line defining a surface.

3. The Fundamental Theorem of Curls

Given a vector field $\vec{v}(x,y,z)$:

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint_{\text{path}} \vec{v} \cdot d\vec{l}$$

This is Stokes' Theorem.

The integral of a derivative (curl) over a region (surface) is equal to the value at the boundary (perimeter path).

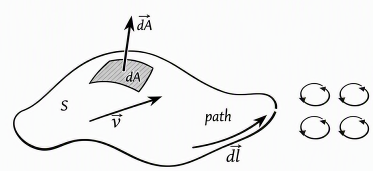

Figure 5: The area integral of the vorticity is just the integral of the vector around the perimeter – i.e. adding up the local vorticity.

Right-Hand Rule for direction of $d\vec{l}$ and $d\vec{a}$: the right hand curls in the direction of the path; the thumb points in the direction of $d\vec{a}$.

Corollary 1: $\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}$ depends only on the boundary line, not the particular surface.

Corollary 2: $\oint (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = 0$ for any closed surface (because the boundary line shrinks to a point). Because $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0 \; \forall \, \vec{v}$.

Spherical Coordinates

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta$$

$$r \in [0, \infty], \qquad \theta \in [0, \pi], \qquad \phi \in [0, 2\pi)$$

(Note: θ ranges over $[0, \pi]$, not 2π. Also, $r \geq 0$.)

Element of volume:

$$d\tau = dx\,dy\,dz = dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi = r^2\,dr\,\sin\theta\,d\theta\,d\phi$$

Note that the unit vectors change direction as the point $\vec{r}$ in spherical coordinates moves! As a result, the gradient, divergence, and curl take on new (and complicated) forms in spherical coordinates. These are given on the inside front cover of the textbook. We express a vector in spherical coordinates using the unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$:

$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$$

Cylindrical Coordinates

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z$$

Element of volume:

$$d\tau = ds\,s\,d\phi\,dz = s\,ds\,d\phi\,dz$$

$$s \in [0, \infty], \qquad \phi \in [0, 2\pi), \qquad z \in (-\infty, \infty)$$

The gradient, divergence, and curl take on special forms for cylindrical coordinates. These are given in the inside front cover of the textbook.

The Dirac Delta Function

This function is very convenient for representing sources of electric charge and current, and is used in many places in theoretical physics.

One dimensional delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

A useful property of the delta function is that it picks out the value of a function at a given point:

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

Generalized 3-Dimensional Delta Function:

$$\delta^3(\vec{r}) = \delta(x)\, \delta(y)\, \delta(z)$$

with

$$\int_{\text{All Space}} \delta^3(\vec{r})\, d\tau = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\ \delta(x)\delta(y)\delta(z) = 1$$

Similarly,

$$\int_{\text{All Space}} f(\vec{r})\, \delta^3(\vec{r}-\vec{a})\, d\tau = f(\vec{a})$$

One of the most useful results in electrodynamics is the divergence of $\hat{r}/r^2$:

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta^3(\vec{r})$$

The vector function $\hat{r}/r^2$ diverges from the origin.

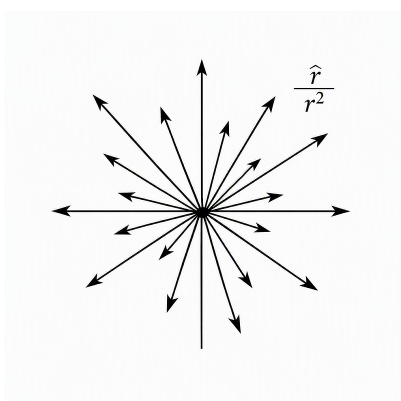

Figure 6: The vector function $\hat{r}/r^2$ diverges outward from the origin. This figure does not capture the magnitude of the function correctly, only the direction.

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) + 0 + 0 = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

except at $r = 0$, where it diverges ($= \infty$ at $r = 0$).

But the Divergence Theorem says:

$$\int_V \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) d\tau = \oint_S \frac{\hat{r}}{r^2} \cdot d\vec{a}$$

Take as the surface S a sphere of radius R centered on the origin:

$$d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$\oint_S \frac{\hat{r}}{r^2} \cdot d\vec{a} = \int_0^{2\pi} \int_0^{\pi} \frac{\hat{r} \cdot \hat{r} \, R^2}{R^2} \sin\theta \, d\theta \, d\phi$$
$$= 2\pi \int_0^{\pi} \sin\theta \, d\theta = 2\pi \left[-\cos\theta \right]_0^{\pi} = -2\pi(-1-1) = 4\pi$$

The correct divergence result is

$$\int_V \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) d\tau = \int_V 4\pi \, \delta^3(\vec{r}) \, d\tau = 4\pi$$

So the divergence theorem is satisfied for this definition of the divergence of $\hat{r}/r^2$. So:

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \, \delta^3(\vec{r})$$

The Helmholtz Theorem

Maxwell's Equations specify the divergence and curl of the electric and magnetic fields.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad\qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Are knowledge of the divergence and curl of a vector function sufficient to determine the function uniquely?

Yes, if the vector function goes to zero as $r \to \infty$, and if its curl and divergence both go to zero faster than $1/r^2$ as $r \to \infty$. See Appendix B for a proof.

This is an extremely important result. It says that if we can solve these 4 equations (self-consistently) for $\vec{E}$ and $\vec{B}$, subject to the stated constraints, then we can solve any problem in electrodynamics.