Lecture #28 Summary
PHYS 313 Electricity and Magnetism I

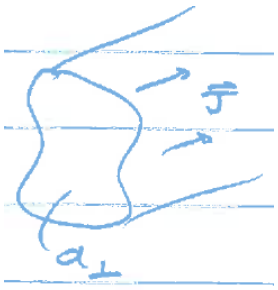
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Fall 2026

Conservation of Charge in Current Flow

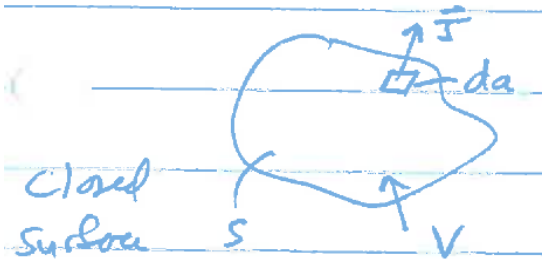
In 3D, the total current is the rate at which charge passes a given plane.

$$I = \frac{\Delta Q}{\Delta t} = \int_S J\, da_\perp$$



$$I = \int_S \vec{J} \cdot d\vec{a}$$

This integral picks out the part of $\vec{J}$ flowing $\perp$ to the area.
Using the divergence theorem, we can apply this expression to a closed surface



$$\oint_S \vec{J} \cdot d\vec{a} = \int_{\mathscr{V}} \vec{\nabla} \cdot \vec{J}\, d\tau \qquad \longrightarrow \text{ Divergence Theorem}$$

This integral represents the rate at which charge is leaving the volume $\mathscr{V}$. Because charge is conserved, this must come at the expense of the total charge enclosed inside the surface

$$\int_{\mathscr{V}} \vec{\nabla} \cdot \vec{J}\, d\tau = -\frac{d}{dt} Q_{\text{enc}} = -\frac{d}{dt} \int_{\mathscr{V}} \rho\, d\tau$$

The total time derivative could have changes due to
1) ρ $\quad \partial\rho/\partial t$

2) $\mathscr{V}$ $\quad \partial\mathscr{V}/\partial t$.

If the volume is constant in time, then we have

$$\int_{\mathscr{V}} \vec{\nabla}\cdot\vec{J}\,d\tau = -\int_{\mathscr{V}} \frac{\partial\rho}{\partial t}\,d\tau$$

This must be true for any arbitrarily defined volume $\mathscr{V}$ so we must assume

$$\boxed{\vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t} \qquad \text{At every point in space.}$$

This is called the continuity equation. It is a statement of local charge conservation.

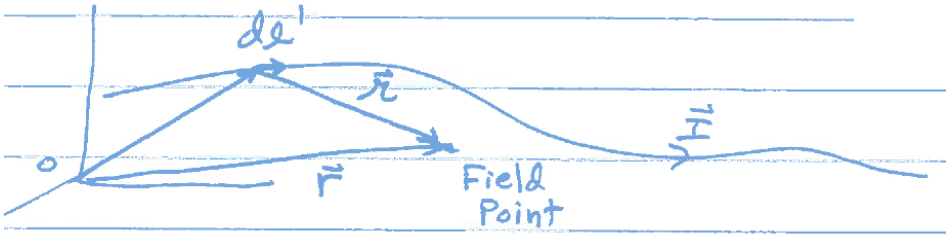
We will now consider steady current flow. In this case charge will not pile up in certain locations, so $\partial\rho/\partial t = 0$. Hence we have

$$\boxed{\vec{\nabla}\cdot\vec{J} = 0}$$

Steady currents produce magnetic fields that are constant in time.

Steady Currents are the Source of Magnetic Field in Magnetostatics

Consider a steady current $\vec{I}$ flowing in a filamentary wire



The Biot-Savart Law

The current produces a magnetic field $\vec{B}$ at a field point $\vec{r}$.

This is the rough analog of Coulomb's Law in Electrostatics. Both are $\sim 1/\mathscr{r}^2$ laws.

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{\mathscr{r}}}{|\mathscr{r}|^2}\,d\ell'}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{d\vec{\ell}'\times\hat{\mathscr{r}}}{|\mathscr{r}|^2}$$

1) One integrates along the current path in the direction of the current.

2) Here μ_0 is the permeability of free space

$$\mu_0 = 4\pi\times 10^{-7}\ \mathrm{N/A}^2 = 4\pi\times 10^{-7}\ \mathrm{H/m} \qquad H = \Omega s = \frac{Js^2}{C^2}$$

3) Units of $\vec{B}$:

$$\vec{F} = q\,\vec{v}\times\vec{B}$$

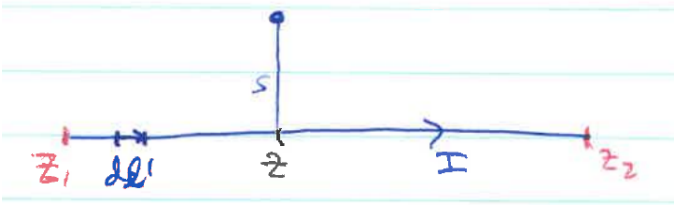
$$N = C\,\frac{m}{s}\,B \;\Rightarrow\; B = \frac{N}{Am} \qquad 1\ \text{Tesla} = 1\,\frac{N}{Am}$$

The units of magnetic field are Tesla = Newton per Amp-meter.

4) The Biot-Savart is linear in $I \Rightarrow$ Linear superposition holds. The magnetic fields produced by multiple currents simply add (as vector quantities).

Example use of the Biot-Savart Law

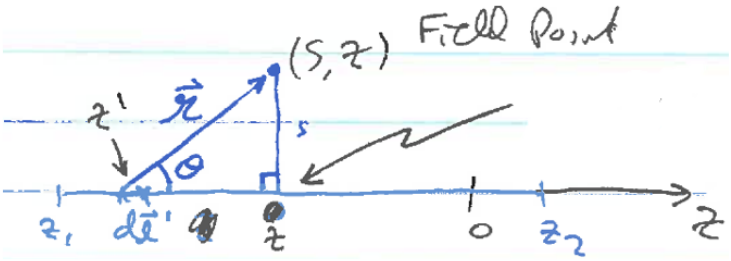
Calculate the magnetic field due to a straight line segment current I a perpendicular distance s from the wire. The wire extends from z_1 to z_2 and the field point is a distance s from location z. The current I flows from left to right.



Utilize cylindrical coordinates (s, z, ϕ)

$$\vec{B}(s,z) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\ell}' \times \hat{\imath}}{\imath^2}$$

Find $d\vec{\ell}'$, $\hat{\imath}$, $\imath^2$ first, then put them together in the integral.



$$d\vec{\ell}' = dz'\,\hat{z}$$

$$\imath^2 = s^2 + (z' - z)^2$$

$$d\vec{\ell}' \times \hat{\imath} = d\ell' \sin\theta\,\hat{\phi} = dz' \sin\theta\,\hat{\phi}$$

So

$$\vec{B}(s,z) = \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} \frac{\sin\theta\,\hat{\phi}\,dz'}{s^2 + (z'-z)^2}$$

But

$$\sin\theta = \frac{s}{\imath} = \frac{s}{\sqrt{s^2 + (z'-z)^2}}$$

$$\vec{B}(s,z) = \frac{\mu_0 I s\,\hat{\phi}}{4\pi} \int_{z_1}^{z_2} \frac{dz'}{\{s^2 + (z'-z)^2\}^{3/2}}$$

Let $u \equiv z' - z$ so $du = dz'$
Then we need to solve this integral:

$$\int \frac{du}{[s^2 + u^2]^{3/2}} = \frac{u}{s^2\sqrt{s^2 + u^2}}$$

so

$$\vec{B}(s,z) = \frac{\mu_0 I s\,\hat{\phi}}{4\pi} \left. \frac{z'-z}{s^2\sqrt{s^2 + (z'-z)^2}} \right|_{z_1}^{z_2}$$

$$= \frac{\mu_0 I\,\hat{\phi}}{4\pi s} \left\{ \frac{z_2 - z}{\sqrt{s^2 + (z_2 - z)^2}} - \frac{z_1 - z}{\sqrt{s^2 + (z_1 - z)^2}} \right\}$$

Now consider the limit:

$$z_2 \to +\infty \qquad z_1 \to -\infty \qquad \text{with } z \text{ finite}$$

Then

$$\vec{B}(s,z) \simeq \frac{\mu_0 I\,\hat{\phi}}{4\pi s} \left\{ \frac{z_2}{\sqrt{z_2^2}} - \frac{z_1}{\sqrt{z_1^2}} \right\}$$

$$\{1-(-1)\} \to 2$$

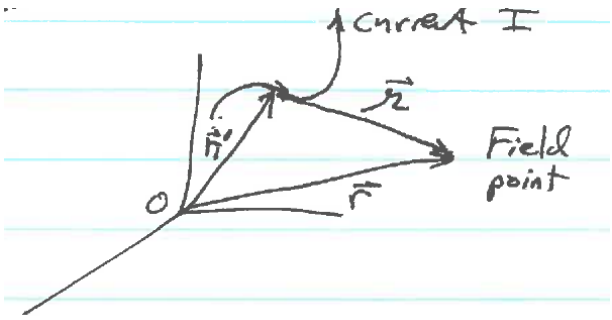
$$\boxed{\vec{B}(s) = \frac{\mu_0 I\,\hat{\phi}}{2\pi s}} \qquad \text{for an infinite straight wire}$$

This is what we expect from Ampere's Law, which is coming up in the next lecture.

This also illustrates the **Right Hand Rule**. Grab the wire with thumb of your right hand pointing in the direction of the current $\Rightarrow$ the fingers curl in the direction of $\vec{B}$.

The Biot-Savart law can be generalized to other types of currents besides filamentary.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\substack{\text{line}\\ \text{current}\\ \text{path}}} \frac{\vec{I} \times \hat{\imath}}{\imath^2}\, d\ell'$$



$$\int \vec{I}\, d\ell' \;\to\; \int \vec{K}\, da' \;\to\; \int \vec{J}\, d\tau'$$

so for surface currents

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\substack{\text{surface}\\ \text{current}}} \frac{\vec{K} \times \hat{\imath}}{\imath^2}\, da'}$$

And for volume current:

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\substack{\text{volume}\\ \text{current}}} \frac{\vec{J} \times \hat{\imath}}{\imath^2}\, d\tau'}$$

However, these integrals can be quite difficult to perform.

$\Rightarrow$ This is reminiscent of the direct calculation of $\vec{E}$ from a given charge distribution!

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\vec{r}\,')\,\hat{\imath}}{\imath^2}\, d\tau'$$

$\Rightarrow$ We will develop easier ways to calculate $\vec{B}$ from a current distribution, similar in spirit to Gauss's Law. But first the preliminaries...

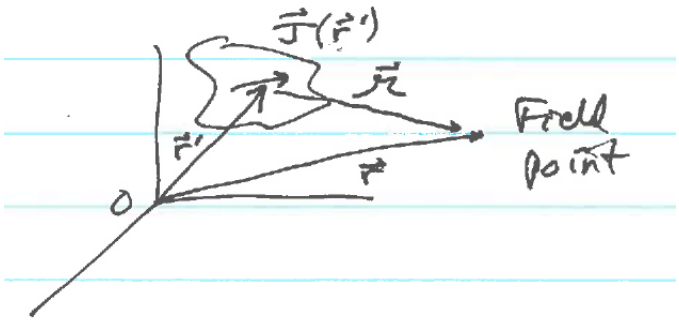
What are the divergence and curl of $\vec{B}$?

Given these two things, and the boundary conditions, we should be able to calculate $\vec{B}$ everywhere in space.

① Divergence of $\vec{B}$:

Start with the 3D version of Biot-Savart

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\vec{J}(\vec{r}\,') \times \hat{\imath}}{\imath^2}\, d\tau'$$



Calculate $\vec{\nabla} \cdot \vec{B}$. The divergence is w.r.t. the field point coordinates ($\vec{r}$), so it runs right into the integral (which is over the source-point coordinates $\vec{r}'$, recall that $\vec{\imath} = \vec{r} - \vec{r}'$):

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \vec{\nabla} \cdot \left(\frac{\vec{J}(\vec{r}') \times \hat{\imath}}{\imath^2} \right) d\tau'$$

Use the product rule on the divergence (#6 in the inside front cover of Griffiths):

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Let $\vec{A} = \vec{J}$ and $\vec{B} = \dfrac{\hat{\imath}}{\imath^2}$
So

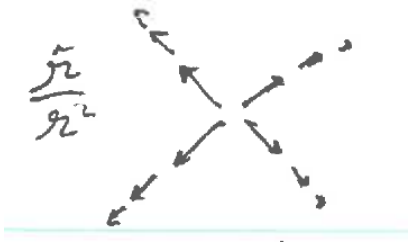
$$\vec{\nabla} \cdot \left(\vec{J} \times \frac{\hat{\imath}}{\imath^2} \right) = \frac{\hat{\imath}}{\imath^2} \cdot (\vec{\nabla} \times \vec{J}) - \vec{J} \cdot \left(\vec{\nabla} \times \frac{\hat{\imath}}{\imath^2} \right)$$

But first $\vec{J} = \vec{J}(\vec{r}')$ and has no $\vec{r}$-dependence, so

$$\vec{\nabla} \times \vec{J}(\vec{r}') = 0$$

Next, the curl of $\dfrac{\hat{\imath}}{\imath^2} = 0$ (unlike the divergence, which we did this in prob 1.62 of HW 2 gives $4\pi\,\delta^3(\vec{\imath})$)

$$\vec{\nabla} \times (r^n\,\hat{r}) = 0 \quad \forall n.$$



Clearly this function has no curl!
Hence

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} 0 \; d\tau'$$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

The magnetic field has no divergence, anywhere in space. This means that there are no monopole sources of magnetic fields. In other words there are no point magnetic charges, otherwise known as magnetic monopoles!

This result has major consequences for magnetics vs. electronics. Point electric charges are shuttled around to perform computations in electronic computers. There are no corresponding magnetic particles that could be used to create an analogous form of "magnetic electronics." Instead, we have to start with the next term in the multi-pole expansion for magnetostatics, namely the dipole term.