Lecture #25 Summary
PHYS 313 Electricity and Magnetism I

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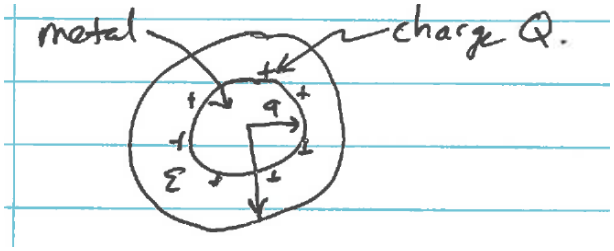
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The issue of self-consistency among $\vec{E}$, $\vec{P}$, $\vec{D}$

Finding the electric field $\vec{E}$, the polarization $\vec{P}$ and the displacement field $\vec{D}$ sometimes has to be done self-consistently. Consider an LIH polarizable object. In zero electric field it has zero polarization. Apply a field and it develops polarization. However, this newly-created polarization then creates its own electric field, both inside and outside the object. This adds to the original electric field, which in turn will create a new polarization. The cycle continues until the changes are too small to make a difference (you will do this calculation yourself in Homework 8). We would like to find an easier way to determine $\vec{E}$, $\vec{P}$ and $\vec{D}$ that is fully self-consistent. The electric displacement $\vec{D}$ is the enabler.

Consider Example 4.5: A metal sphere of radius a carries a charge Q. It is surrounded by an LIH dielectric material of permittivity ϵ, out to radius $b > a$.

Find the potential V at the center of the sphere, relative to $V = 0$ at infinity.



1) To find V, we need $\vec{E}$, so that we can use

$$V(\vec{r}) = -\int_{\text{Ref. point}}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

2) To find $\vec{E}$ we need to know ρ_f and ρ_b to utilize $\nabla \cdot \vec{E} = \rho/\epsilon_0$. But to get $\rho_b = -\nabla \cdot \vec{P}$, we need to know $\vec{P}$.

3) But we need $\vec{E}$ to get $\vec{P}$ since $\vec{P} = \epsilon_0 \chi_e \vec{E}$

This seems like an impossible task!

Here the displacement field $\vec{D}$ comes to the rescue.

We know there is free charge Q and can assume that it is uniformly distributed over the surface of the metal sphere.

$\Rightarrow$ Use Gauss' law for $\vec{D}$

$$\oint \vec{D} \cdot d\vec{a} = Q_{\text{free,enc}}$$

Take $\vec{D}(\vec{r}) = |\vec{D}(\vec{r})|\,\hat{r}$ inside the dielectric. Then

$$|\vec{D}|\,\hat{r} \cdot \hat{r}\, 4\pi r^2 = Q$$

Then

$$\vec{D}(r) = \frac{Q}{4\pi r^2}\,\hat{r} \qquad \text{for } r > a$$

Inside the metal sphere $\vec{E} = 0$ so $\vec{D} = 0$, so $\vec{P} = 0$.

Now we can find $\vec{E}$ from $\vec{D}$: $\vec{D} = \epsilon\vec{E}$, so

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\,\hat{r} \qquad a < r < b \quad \text{inside the dielectric}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r} \qquad r > b \quad \text{in free space outside}$$

$$\vec{E} = 0 \qquad r < a \quad \text{inside the conductor}$$

Now we can evaluate the potential at the center of the sphere

$$V(0) = -\int_\infty^0 \vec{E}\cdot d\vec{l} = -\int_\infty^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr - \int_b^a \frac{Q}{4\pi\epsilon r^2}\,dr - \int_a^0 0\,dr$$

Note that this is a purely radial integral from ∞ to 0.

$$V(0) = \frac{Q}{4\pi\epsilon_0}\,\frac{1}{r}\bigg|_\infty^b + \frac{Q}{4\pi\epsilon}\,\frac{1}{r}\bigg|_b^a + 0$$

$$\boxed{V(0) = \frac{Q}{4\pi}\left\{\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right\}} \qquad \text{This answers the question.}$$

To solve this problem it was not necessary to find the polarization $\vec{P}$, but we can do it now.
Inside the dielectric

$$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\,\hat{r}$$

What is the bound charge $\rho_b = -\vec{\nabla}\cdot\vec{P}$?
$\vec{P}$ has only a radial coordinate, so there is only one term in the divergence in spherical coordinates,

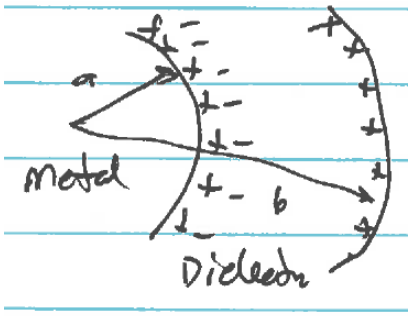
$$\vec{\nabla}\cdot\vec{P} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\right) = 0 \qquad \text{No volume bound charge } \rho_b = 0.$$

However, there is surface bound charge $\sigma_b = \vec{P}\cdot\hat{n}$
There are 2 surfaces on the dielectric object, the one at $r = a$ (inner) and $r = b$ (outer)
Outer surface $r = b$, and outward normal direction $\hat{n} = \hat{r}$

$$\boxed{\sigma_b(r = b) = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}}$$

Inner surface $r = a$, and outward normal vector $\hat{n} = -\hat{r}$

$$\boxed{\sigma_b(r = a) = \frac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}}$$



The dielectric reduces the electric field inside itself by inducing charges to partially absorb the electric field lines. A dielectric is an imperfect metal, in some sense.

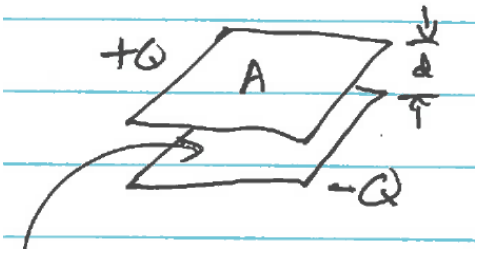
Note that the total induced surface charge on the dielectric object is zero

$$Q_{\text{induced}} = 4\pi a^2\left(\frac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}\right) + 4\pi b^2\left(\frac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}\right) = 0$$

Dielectric-Filled Capacitor

A capacitor has its capacitance C enhanced by filling the gap between the two conductors with a dielectric material.

Consider the parallel plate capacitor of plate area A and separation d holding charges $+Q$ and $-Q$ on the two plates:



Add a uniform LIH dielectric material into the space between the plates.

The electric field in the gap is reduced to

$$\vec{E} \to \frac{1}{\epsilon_r}\vec{E}$$

by the relative dielectric constant ϵ_r; $\epsilon = \epsilon_r \epsilon_0$.

As a result, the potential difference between the plates, for a given charge Q, is reduced by a factor of ϵ_r

$$V \to \frac{1}{\epsilon_r}V$$

The capacitance C is thus enhanced:

$$\boxed{C = \frac{Q}{V} \to \frac{Q}{V/\epsilon_r} = \epsilon_r \frac{Q}{V}}$$

There are materials that are highly polarizable, having $\epsilon_r \gg 1$. These can be used to greatly enhance the capacitance without changing the physical size of the capacitor.

Bound and Free charge in a Linear Dielectric

Consider a homogeneous linear and isotropic dielectric (LIH = Linear, Isotropic, Homogeneous).

The bound charge density is

$$\rho_b = -\nabla \cdot \vec{P}$$

But

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

and

$$\vec{D} = \epsilon \vec{E}, \text{ so}$$
$$\vec{P} = \frac{\epsilon_0}{\epsilon}\chi_e \vec{D}$$

Now we have

$$\begin{aligned}\rho_b = -\nabla \cdot \vec{P} &= -\nabla \cdot \left(\frac{\epsilon_0}{\epsilon}\chi_e \vec{D}\right) \\ &= -\frac{\epsilon_0}{\epsilon}\chi_e \nabla \cdot \vec{D} \qquad \text{(Recall that } \nabla \cdot \vec{D} = \rho_f\text{)}\end{aligned}$$

so

$$\begin{aligned}\rho_b &= -\frac{\epsilon_0}{\epsilon}\chi_e \rho_f \\ &= -\frac{\epsilon_0 \chi_e}{\epsilon_0 (1+\chi_e)}\rho_f\end{aligned}$$

$$\boxed{\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f} \quad \text{in a LIH dielectric.}$$

Hence the existence of bound charge density requires that there be free charge density inside the dielectric material. Usually this is not the case, and we often have

$$\rho_f = 0 \Rightarrow \rho_b = 0 \Rightarrow \rho = \rho_f + \rho_b = 0$$

As a result, any free or bound charge must be at the surface of the LIH dielectric.

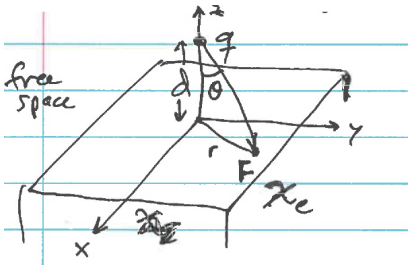
Since $\rho = 0$, $\nabla \cdot \vec{E} = \rho/\epsilon_0 = 0$ and $\nabla^2 V = 0$, hence all of the methods to solve Laplace's equation also work in LIH dielectrics with $\rho = 0$. These methods include:

Method of Images, and

Separation of Variables (cartesian, cylindrical, spherical)

Method of Images with Dielectrics

Consider Example 4.8: A point charge q is situated a distance d above a semi-infinite LIH dielectric half-space, with susceptibility χ_e. What is the force acting on the point charge?



We solved this problem before for a charge above a conducting half-space, using the method of images. Now we want to use the same method for an *imperfect* dielectric.

The charge q will attract charges of the opposite sign on the surface, hence there will be an attractive interaction.

We also know that there is no free or bound charge in the dielectric: $\rho_b = \rho_f = \rho = 0$. Hence we can use the method of images to solve Laplace's equation.

To calculate the force, we have to calculate the surface bound charge density

$$\sigma_b = \vec{P} \cdot \hat{n} = P_z = \epsilon_0 \chi_e E_z$$

where E_z is the *total* electric field just inside the dielectric, produced by all charges, free and bound. Note that $\sigma_b(x, y)$ and $E_z(x, y)$ will not be uniform, but instead are peaked just below the charge q.

How to find E_z? This is where it gets interesting ...

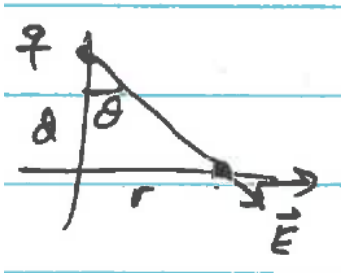
E_z has two contributions:

$$\begin{cases} \vec{E} \text{ due to } q \\ \vec{E} \text{ due to } \sigma_b \end{cases}$$

Calculate each of these contributions, and then add them up.

1) From Coulomb's law, and the above diagram, we can find E_z at the field point F on the surface of the dielectric

$$\vec{E}(F) \cdot \hat{z} = -\frac{1}{4\pi\epsilon_0} \frac{q}{(d^2 + r^2)} \cos\theta$$



But

$$\cos\theta = \frac{d}{\sqrt{d^2 + r^2}}$$

so,

$$E_z(F) = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(d^2 + r^2)^{3/2}}$$

Aside: Why isn't the field produced by the point-charge, evaluated in the dielectric, divided by ϵ?

Because the factor $1/\epsilon$ is the result of bound charge screening, and including both would count the screening twice.

In Griffiths' ϵ_0-based formulation, the electric field $\vec{E}$ is always the vacuum Coulomb field summed over all charges, free and bound. The point charge q produces field $q/(4\pi\epsilon_0 r^2)$ at any point, whatever medium surrounds it. Its field doesn't know about the dielectric. The dielectric acts only through the bound charges it develops, and those produce their own fields that partially cancel charge q's field inside the material. That cancellation is the "reduction" you expect, and here it appears as the $-\sigma_b/2\epsilon_0$ term calculated next.

2) By symmetry, a charge distribution confined to the plane $z = 0$ produces a z-field that is odd across the plane, $+\sigma_b/2\epsilon_0$ just above and $-\sigma_b/2\epsilon_0$ just below. The z-component of $\vec{E}$ from the bound charge is

$$E_{zb} = \frac{-\sigma_b}{2\epsilon_0} \qquad \text{in the dielectric below the surface}$$

Now use the two E_z contributions calculated above to evaluate $\sigma_b = \epsilon_0\chi_e E_z$

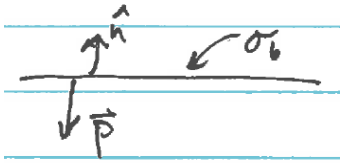
$$\sigma_b = \epsilon_0\chi_e\left\{-\frac{1}{4\pi\epsilon_0}\frac{qd}{(d^2 + r^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0}\right\}$$

Solve for σ_b

$$\sigma_b\left(1 + \frac{\chi_e}{2}\right) = -\frac{\chi_e}{4\pi}\frac{qd}{(d^2 + r^2)^{3/2}}$$

so

$$\boxed{\sigma_b = \frac{-1}{2\pi}\frac{\chi_e}{2 + \chi_e}\frac{qd}{(r^2 + d^2)^{3/2}}}$$



Since we find that $\vec{E}$ is pointing downwards, it must be that $\vec{P}$ is also pointing downwards, hence $\sigma_b = \vec{P}\cdot\hat{n} < 0$, consistent with the result above for the sign of the bound surface charge density.

This is very similar to the result for the (free) surface charge on a conductor:

$$\sigma(x, y) = \frac{-qd}{2\pi\,(r^2 + d^2)^{3/2}} \qquad \text{Eq. (3.10)} \qquad r^2 = x^2 + y^2$$

except for the factor $\dfrac{\chi_e}{2 + \chi_e}$ that appears for the dielectric case.

We can deduce that the total bound charge is

$$\boxed{q_b = -q\left(\frac{\chi_e}{2 + \chi_e}\right)}$$

Note that in the limit $\chi_e \to \infty$, this gives $q_b = -q$, which is what we expect for a perfect dielectric, in other words, a conductor in electrostatic equilibrium.

Next step: Take this $\sigma_b(x, y)$ distribution and calculate the force of interaction with q.

We could use the electric field calculation

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\imath}}{\imath^2}\,\sigma_b(\vec{r}\,')\,da'$$

to find the field at the location of q, and then get the force.
Instead, we can use the Method of Images.

We can place an image charge q_b at $z = -d$ to get for the potential at $z > 0$

$$V(x,y,z>0) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right\}$$

One can put the charge q_b at $z = +d$ to get the potential inside the dielectric ($z < 0$):

$$V(x,y,z<0) = \frac{1}{4\pi\epsilon_0}\,\frac{q+q_b}{\sqrt{x^2+y^2+(z-d)^2}}$$

This is an inspired guess, based on prior experience with the conductor case. However, it is correct because,
i) $V(x,y,z)$ satisfies Laplace's equation,
ii) $V \to 0$ as $x, y, z \to \infty$,
iii) $V(z)$ is continuous at $z = 0$,
iv) dV/dz is discontinuous at $z = 0$ due to the bound surface charge density.
Hence it must be correct.
The force on the charge q is just that due to the image charge q_b, and the force is attractive,

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q\,q_b}{(2d)^2} = -\frac{1}{4\pi\epsilon_0}\,\frac{\chi_e}{2+\chi_e}\,\frac{q^2}{4d^2}\,\hat{z}}$$

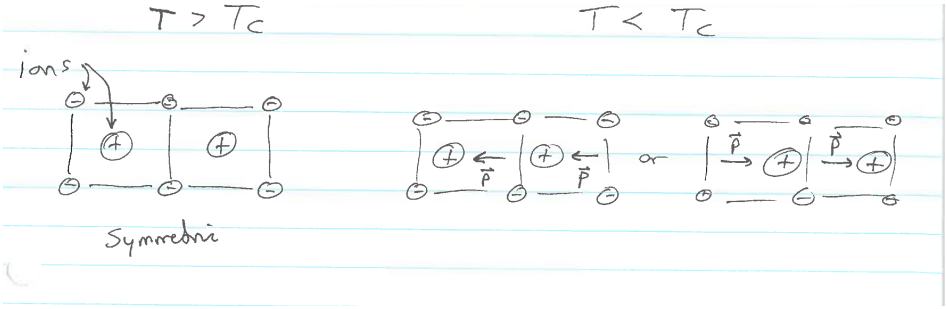
Ferroelectricity

Some materials develop a spontaneous non-zero macroscopic polarization vector $\vec{P}$ even in the absence of an applied field. (They are **not** LIH) These are called "Ferroelectric" materials.

The name is unfortunate because most of these materials have no Iron (Ferrous or Fe) materials in them.

They get the name from being analogous to Ferromagnets, which develop a non-zero magnetization (magnetic moment per unit volume) in the absence of an external magnetic field.

Ferroelectrics:
These materials develop a polarization $\vec{P}$ by moving ions about in the unit cell of the material below a Curie Temperature T_C.

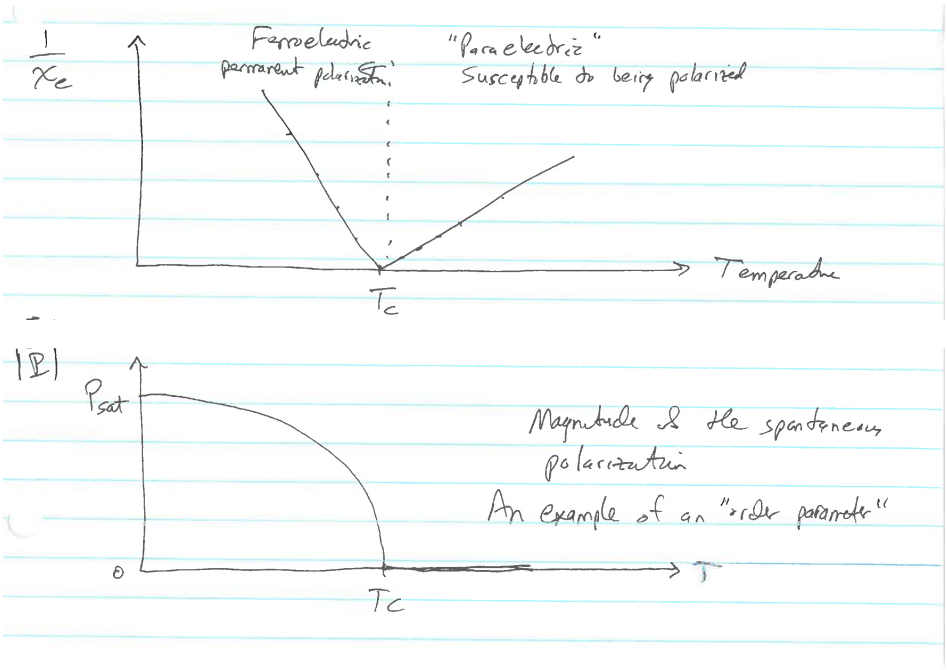


At high temperature, thermal agitation prevents the ionic displacements from aligning and producing a net polarization. In this state they have $\vec{P} = 0$.

As the temperature is reduced the displacements can occur more frequently and they have better long-range correlations.

At the Curie temperature, the electric susceptibility of the solid diverges.

$$\boxed{\chi_e \sim \frac{\chi_o}{T - T_C} \qquad T > T_C}$$ Curie-Weiss Law



Material	T_C (K)	P_{sat} (C/m^2)	
$BaTiO_3$	408	0.26	(296K)
$LiNbO_3$	1480	0.71	(296K)
Tri-Glycene Sulfate	322	0.03	(29K)
KH_2AsO_4	97	0.05	(78K)

The Curie temperatures vary considerably depending on the details of the ions and their crystal structure.

The saturation polarizations $P_{sat} = P(T = 0)$ are all on the order of 1 C/m^2, or less.

$LiNbO_3$ is a nonlinear dielectric and used extensively in nonlinear optics driven by intense laser pulses. It has a nonlinear dielectric susceptibility:

$$\boxed{\vec{P} = \epsilon_o \chi \vec{E}}$$

with

$$\chi = \chi_1 + \chi_2 |\vec{E}| + \chi_3 |\vec{E}|^2 + \cdots$$

so

$$\boxed{\vec{P} = \epsilon_o \Big(\chi_1 \vec{E} + \underbrace{\chi_2 |\vec{E}| \vec{E} + \chi_3 |\vec{E}|^2 \vec{E} + \cdots}_{\text{Generate harmonics, mixing.}} \Big)}$$

A green laser pointer starts by generating intense light in the infrared at a wavelength of 1064 nm. The 1064 nm infrared light then passes through a nonlinear frequency-doubling crystal, usually made of KTP (potassium titanyl phosphate, $KTiOPO_4$). This crystal has a strong χ_2 and halves the wavelength, turning the 1064 nm infrared light into visible 532 nm green light.