Lecture #24 Summary
PHYS 313 Electricity and Magnetism I

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Fall 2026

Gauss's Law & Electric Displacement $\vec{D}$

How is Gauss's Law modified in the presence of a dielectric?
Inside a dielectric material, there can be free charge density and bound charge density,

$$\rho(\vec{r}) = \rho_f(\vec{r}) + \rho_b(\vec{r})$$

Free Charge – That which is not the result of polarization. One could connect a wire to it and drain it off to a lower potential.

Then

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_f + \rho_b \qquad \text{where } \vec{E} \text{ is the total electric field}$$

But

$$\rho_b = -\vec{\nabla} \cdot \vec{P}(\vec{r}) \qquad \text{where } \vec{P} \text{ is the polarization of the dielectric medium}$$

(this holds inside the dielectric – surface charge is handled as a boundary condition where $\hat{n} \cdot \vec{P}$ changes discontinuously). Hence,

$$\vec{\nabla} \cdot \left(\epsilon_0 \vec{E} + \vec{P} \right) = \rho_f$$

Electric Displacement $\vec{D}$ is defined as,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

and

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f} \qquad \text{Gauss's Law in a dielectric medium}$$

or in integral form: $\oint d\vec{a} \cdot \vec{D} = Q_f$, where Q_f is the total enclosed *free* charge.

In other words, $\vec{D}$ can be calculated from the free charge in a dielectric. To get $\vec{E}$ from $\vec{D}$, we need to know something about the polarization $\vec{P}$.

Gauss's law in a dielectric medium

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V (\rho_f + \rho_b)\, d\tau$$

$$\rho_b = -\nabla \cdot \vec{P}$$

$$\int_V \frac{\rho_b}{\varepsilon_0}\, d\tau = -\frac{1}{\varepsilon_0} \int \nabla \cdot \vec{P}\, d\tau = -\frac{1}{\varepsilon_0} \oint_S \vec{P} \cdot d\vec{a} \qquad \text{(divergence theorem)}$$

so

$$\oint_S \left(\vec{E} + \frac{\vec{P}}{\varepsilon_0} \right) \cdot d\vec{a} = \frac{1}{\varepsilon_0} \int \rho_f\, d\tau = \frac{Q_{enc,f}}{\varepsilon_0}$$

Multiply both sides by ε_0, giving

$$\boxed{\oint_S \vec{D} \cdot d\vec{a} = Q_{enc,f}} \qquad \text{Gauss's Law for the electric Displacement}$$

$Q_{enc,f}$ is the free charge enclosed by the surface.

Because we have recovered Gauss's Law, it would seem that the electric displacement will also satisfy Coulomb's Law

$$\vec{D}(\vec{r}) = \frac{1}{4\pi\epsilon}\frac{q}{\imath^2}\,\hat{\imath} \qquad \epsilon : \text{dielectric constant of the medium.}$$

However, we must also specify the curl of $\vec{D}$ to uniquely determine the vector function. Look at the curl:

$$\nabla \times \vec{D} = \varepsilon_0 \nabla \times \vec{E} + \nabla \times \vec{P}$$

$$= 0 + \nabla \times \vec{P} \qquad \text{in electrostatics}$$

Comparison between electric field and electric displacement:

$\vec{E}$	$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$
$\nabla \cdot \vec{E} = \rho/\varepsilon_0$	$\nabla \cdot \vec{D} = \rho_{free}$
$\oint_S \vec{E} \cdot d\vec{a} = \dfrac{Q_{enc}}{\varepsilon_0}$	$\oint_S \vec{D} \cdot d\vec{a} = Q_{enc,free}$
$\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \rho(\vec{r}\,')\frac{\hat{\imath}}{\imath^2}\,d\tau'$	$\vec{D}(\vec{r}) \neq \dfrac{1}{4\pi}\displaystyle\int \rho_{free}(\vec{r}\,')\frac{\hat{\imath}}{\imath^2}\,d\tau'$

Why not use this form of Coulomb's Law to calculate $\vec{D}$ directly? Treat $\vec{D}$ as having a source of only ρ_f.

However, we need to know both $\nabla \cdot \vec{D}$ and $\nabla \times \vec{D}$.

$\nabla \times \vec{E} = 0$ in electrostatics. What about $\nabla \times \vec{D}$?

$$\nabla \times \vec{D} = -\varepsilon_0 \nabla \times \vec{E} + \nabla \times \vec{P}$$

$$\neq 0 \text{ in general!}$$

Assume the equation for $\vec{D}(\vec{r})$ above is correct (which it is not). Take its curl (Prob. 2.20)

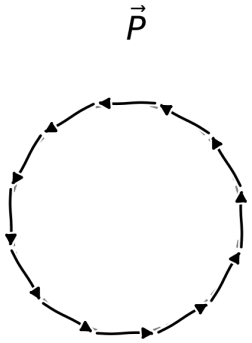
$$\nabla \times \vec{D} = \frac{1}{4\pi}\int \rho_{free}(\vec{r}\,')\,\nabla \times \left(\frac{\hat{\imath}}{\imath^2}\right) d\tau'$$

$$= 0 \quad \text{always, because} \quad \nabla \times \left(\frac{\hat{\imath}}{\imath^2}\right) = 0 \qquad \text{(Prob. 1.63)}$$

However, this contradicts the fact that $\nabla \times \vec{P}$ can be non-zero, showing that $\vec{D}(\vec{r})$ cannot be obtained directly by integrating over just the free charge.

Consequences of $\nabla \times \vec{P} \neq 0$

What about the curl of $\vec{P}$? It need not be zero. For example, a polarized object like this has non-zero curl



This is a possible configuration of polarized charge giving $\nabla \times \vec{P} \neq 0$.

Hence $\nabla \times \vec{D} \neq 0$ in this case, and one **cannot** have

(1) $\oint_{Line} \vec{D} \cdot d\vec{\ell} = 0$

(2) $\int_{\vec{a}}^{\vec{b}} \vec{D} \cdot d\vec{\ell}$ independent of path from $\vec{a}$ to $\vec{b}$

(3) Define a scalar potential function W such that $\vec{D} = -\nabla W$

Clearly $\vec{D}$ does not behave the same way as $\vec{E}$, and so we should not treat it like $\vec{E}$. There is no direct-integration form of Coulomb's Law for $\vec{D}$:

$$\boxed{\vec{D} \neq \frac{1}{4\pi} \int_V \frac{\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} \rho_{free}(\vec{r}\,')\, d\tau'}$$

The properties of $\vec{E}$: $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ and $\nabla \times \vec{E} = 0$ (statics) are determined very generally. The electric displacement $\vec{D}$ includes a specific property of materials ($\vec{P}$), and materials have properties that come in an amazing variety of forms.

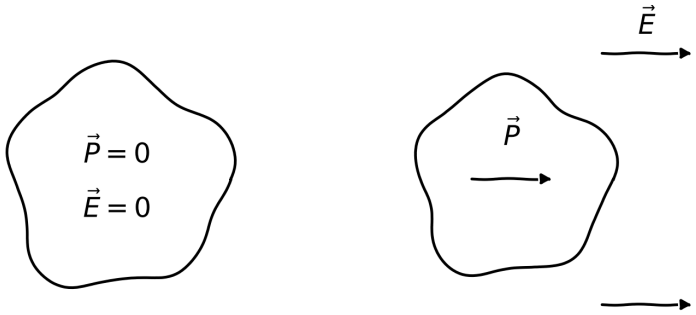
Linear Isotropic Homogeneous (LIH) Dielectrics

Where does polarization come from? For individual atoms, we found

$$\vec{p} = \alpha \vec{E}$$

where α is called the "polarizability" of the atom.

This simple behavior often (but not always) extends to the macroscopic level. In many materials the total polarization of an object is proportional to the applied field.



For these simple linear dielectrics, one can write

$$\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E}} \qquad \text{Linear, isotropic Dielectrics Only}$$

where χ_e is called the electric susceptibility of the object. Its value depends on the material. Some objects are more susceptible to being polarized than others, just as some atoms are more susceptible than others.

Go back to the definition of the displacement field

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$= \varepsilon_0 (1 + \chi_e) \vec{E}$$

$$\equiv \varepsilon \vec{E}$$

Here $\varepsilon \equiv \varepsilon_0(1 + \chi_e)$ is called the permittivity, or dielectric constant, of the material. The more susceptible a material is to being polarized by an electric field, the higher is its permittivity.

Material	Dielectric Constant (20°C)
Vacuum	1
Air (dry)	1.00054
Sapphire	~ 11.1
$SrTiO_2$	230

The Relative Permittivity or dielectric constant is defined as

$$\boxed{\varepsilon_r \equiv \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e}$$

Note that the dielectric "constant" can be a strong function of frequency and temperature.

Dielectric constants generally decrease as a function of increasing frequency. Why?

As the frequency increases, the polarizable degrees of freedom freeze out progressively. The slow processes that involve substantial motion of atoms and molecules freeze out first. They are followed by the next most complicated charge displacements. Basically, the larger the inertia of the objects that are moving under the influence of an alternating electric field, the sooner they effectively stop moving as the frequency increases.

How about the temperature dependence? Thermal agitation tends to destroy the alignment of neighboring dipoles, decreasing the polarization at a given applied field. Decreasing the temperature will allow the dipoles to align better. The example of $SrTiO_3$ is interesting. It appears to undergo a transition to a permanently aligned state, but it is frustrated by quantum mechanical zero point fluctuations. This idea was proposed by K. Alex Muller, who later discovered the high-temperature superconductors.

The total polarizability of a solid has contributions from several sources

(1) Atomic Polarizability $\quad \vec{p} = \alpha \vec{E}$

Shift of electron cloud relative to nucleus $\qquad d \approx 10^{-15}$ m

(2) Ionic Polarizability

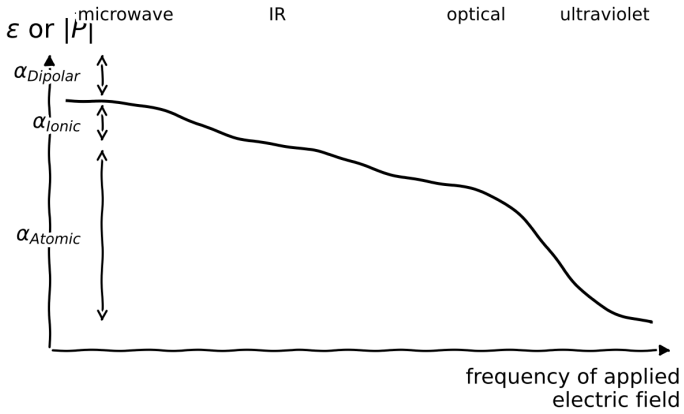
The ions in a solid can create polarization by shifting from their equilibrium positions

$$\begin{array}{c} \mathrm{Cl}^- \ \mathrm{Na}^+ \ \mathrm{Cl}^- \ \mathrm{Na}^+ \ \mathrm{Cl}^- \ \mathrm{Na}^+ \\ \vec{E} \rightarrow \quad \mathrm{Cl}^- \ \mathrm{Na}^+ \ \mathrm{Cl}^- \ \mathrm{Na}^+ \ \rightarrow \vec{E} \qquad d \approx 10^{-10} \text{ m} \end{array}$$

(3) Dipolar Polarizability

Molecules with an intrinsic dipole moment (H_2O) can rotate and align to produce a macroscopic polarization.

One can see these different contributions to the polarization by looking at the frequency-dependence of the relative dielectric constant.



$$\vec{P} = \alpha\vec{E} \qquad \alpha = \alpha_{Atomic} + \alpha_{Ionic} + \alpha_{Dipolar} + \cdots$$

As the frequency of the driving electric field increases, the contributions with the most inertia no longer participate. Eventually one is left with only the atomic polarizability.