

Lecture #21 Summary

PHYS 313 Electricity and Magnetism I

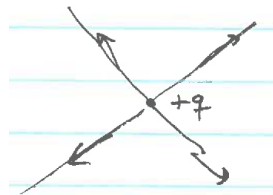
Prof. Steven Anlage

Fall 2026

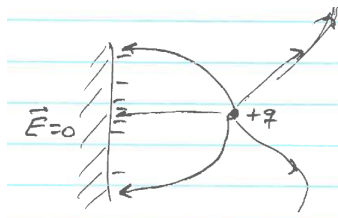
Continuing the calculation of potential and fields outside of a polarized object with zero monopole moment, from Lecture #19...

Once an object becomes polarized, what kind of electric field does it produce inside and outside itself?

In the case of a perfect dielectric (conductor in EE), we found that $\vec{E} = 0$, $V = \text{constant}$, inside a bulk conductor and that the field outside was strongly influenced by the induced charges on the surface.



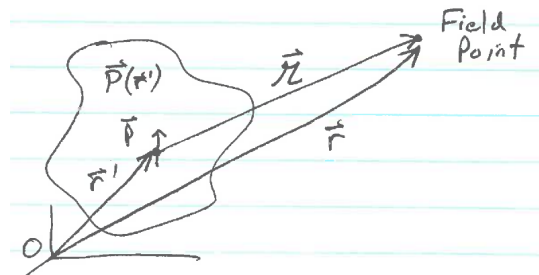
charge in free space



charge next to a grounded conductor
 $\Rightarrow$ Method of images

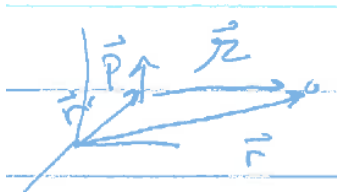
Now let's study the case of a regular dielectric material with finite ϵ_r to see what happens to the electric field inside and outside a polarized object.

Given a polarized material with known polarization $\vec{P}$, what fields does it produce around and inside itself?



The object is assumed to be charge neutral, but polarized.

Chop the material into a whole bunch of elementary dipoles $\vec{p}$. Each produces a potential given by,



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{z}}{z^2}$$

The potential of an elementary dipole falls like $1/z^2$, and the field falls like $1/z^3$
 To construct the total potential due to a macroscopic polarized object, integrate

$$\vec{p} \Rightarrow \vec{P} d\tau'$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\hat{z} \cdot \vec{P}(\vec{r}') d\tau'}{z^2}$$

Note that this is different from the $\vec{E}$ and V calculations that we have done before on monopole charge!

This integral can be transformed into something more useful as follows

Note that

$$\frac{\hat{z}}{z^2} = \vec{\nabla}' \left(\frac{1}{z} \right)$$

$\vec{\nabla}'$ means gradient w.r.t. the source point coordinates $\vec{r}'$

Check: What is $\vec{\nabla}' \left(\frac{1}{z} \right)$?

Do it in Cartesian

$$z = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = |\vec{r} - \vec{r}'|$$

$$\begin{aligned} \vec{\nabla}' \left(\frac{1}{z} \right) &= \frac{\partial}{\partial x'} [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2} \hat{x}' \\ &\quad + \frac{\partial}{\partial y'} [\]^{-1/2} \hat{y}' + \frac{\partial}{\partial z'} [\]^{-1/2} \hat{z}' \\ &= -\frac{1}{2} \frac{2(x - x')}{z^3} \hat{x}' + (y - y') [\]^{-3/2} \hat{y}' + (z - z') [\]^{-3/2} \hat{z}' \\ &= + \frac{\vec{z}}{z^3} = \frac{\hat{z}}{z^2} \end{aligned}$$

$$\vec{\nabla}' \left(\frac{1}{z} \right) = \frac{\hat{z}}{z^2}$$

Also, note that if the gradient is with respect to the *field* point, there is a minus sign:

$$\vec{\nabla} \left(\frac{1}{z} \right) = -\frac{\hat{z}}{z^2}$$

Back to the potential calculation for the polarized material:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \vec{P}(\vec{r}') \cdot \vec{\nabla}' \left(\frac{1}{z} \right) d\tau'$$

Now use product rule #5 from the inside front cover of the textbook:

$$\vec{A} \cdot (\vec{\nabla} f) = \vec{\nabla} \cdot (f \vec{A}) - f(\vec{\nabla} \cdot \vec{A}) \quad \text{where } \vec{A} = \vec{P} \quad \text{and} \quad f = \frac{1}{z}$$

as

$$\vec{P} \cdot \vec{\nabla}' \left(\frac{1}{z} \right) = \vec{\nabla}' \cdot \left(\frac{\vec{P}}{z} \right) - \frac{1}{z} (\vec{\nabla}' \cdot \vec{P})$$

to get

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{V}} \vec{\nabla}' \cdot \left(\frac{\vec{P}}{z} \right) d\tau' - \int_{\mathcal{V}} \frac{1}{z} \vec{\nabla}' \cdot \vec{P} d\tau' \right]$$

Use the divergence theorem on the first term:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\oint_S \frac{1}{z} \vec{P} \cdot d\vec{a}' - \int_{\mathcal{V}} \frac{1}{z} \vec{\nabla}' \cdot \vec{P} d\tau' \right]$$

and S is the surface of the polarized object which occupies the volume $\mathcal{V}$.

Now note that these look like potentials due to charges because they fall off like $1/z$ rather than the $1/z^2$ we started with!

The potential due to a polarized object is that due to a “new” kind of charge - “bound charge”

Look at the two integrands carefully

(1) Surface Integral $\frac{1}{z} \vec{P} \cdot d\vec{a}' = \frac{1}{z} \vec{P} \cdot \hat{n} da'$

Define a bound surface charge density as

$$\sigma_b = \vec{P} \cdot \hat{n}$$

$\hat{n}$ = outward normal vector to surface. See an example of surface bound charge in the Lecture 19 slides.

This is the bound surface charge, or charge per unit area arising at the surface of the polarized object.

The first term is the potential produced by this surface charge density σ_b .

(2) Volume Integral $\frac{1}{z} \vec{\nabla}' \cdot \vec{P}(\vec{r}') d\tau'$

Define

$$\rho_b \equiv -\vec{\nabla} \cdot \vec{P}$$

as the *bound* charge per unit volume inside a polarized object. See an example of volume bound charge in the Lecture 19 slides.

The second term is the potential due to the bound charge inside the object (we are using the fact that $\vec{\nabla} = -\vec{\nabla}'$).

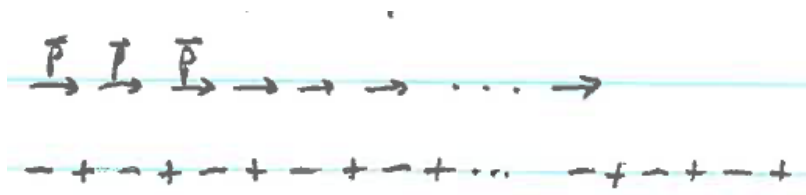
We can re-write the potential as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_b(\vec{r}')}{z} da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b(\vec{r}')}{z} d\tau'$$

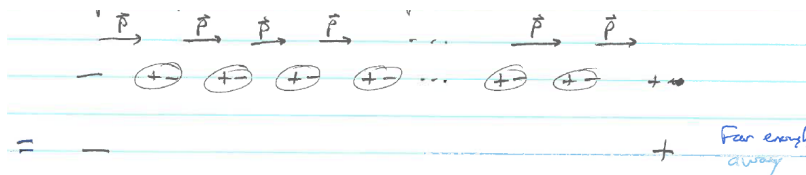
where $\mathcal{V}$ is the volume of the polarized object and S is its surface. This reduces the problem to one that we have solved before - the potential due to surface and volume charges.

Where do these bound charges come from?

Consider a uniformly polarized 1D dielectric

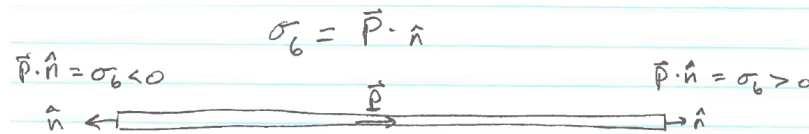


We can pair up the interior charges

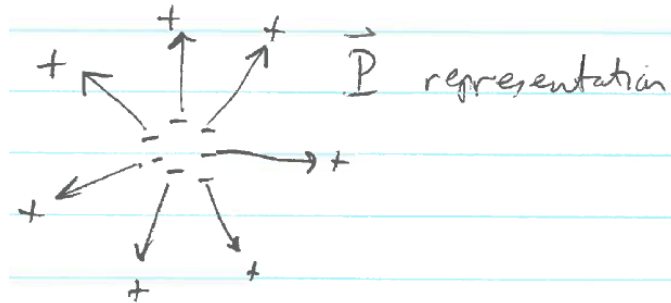


There is a net $-$ charge at one end and a net $+$ charge at the other. These are the bound surface charges described by σ_b .

$$\sigma_b = \vec{P} \cdot \hat{n}$$

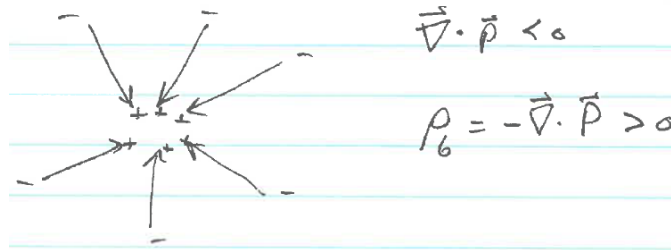


The volume bound charge density ρ_b is due to a divergence of $\vec{P}$. Consider a region of the sample with positive divergence of $\vec{P}$



$$\vec{\nabla} \cdot \vec{P} > 0$$

$\rho_b = -\vec{\nabla} \cdot \vec{P}$ is clearly negative at the part of greatest divergence of $\vec{P}$.
Similar results are expected for a convergence of $\vec{P}$

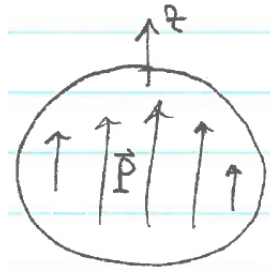


$$\vec{\nabla} \cdot \vec{P} < 0 \quad \rho_b = -\vec{\nabla} \cdot \vec{P} > 0.$$

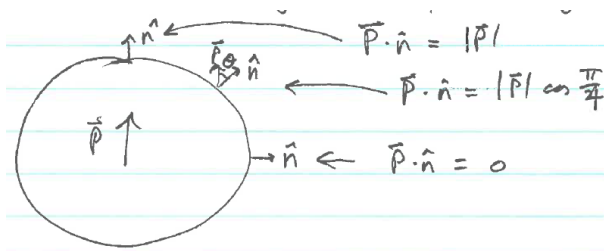
Examples of bound charge in a polarized object

1) Uniformly Polarized Sphere

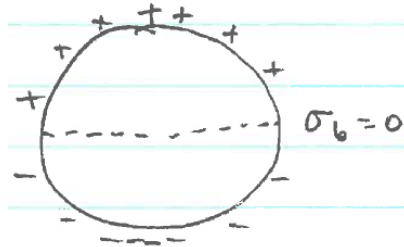
Consider a uniformly polarized sphere of radius R . (This is like a droplet of H_2O in the atmosphere that has been polarized by a lightning bolt.) The polarization field $\vec{P}$ in the sphere is shown in the diagram below. What is the potential and electric field inside and outside it? The first step is to calculate the bound charge.



- Volume charge density $\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$ since the polarization is assumed uniform.
- Surface bound charge density $\sigma_b = \vec{P} \cdot \hat{n}$



So clearly $\sigma_b = \vec{P} \cdot \hat{n} = |\vec{P}| \cos \theta$, where the z -axis is vertical in the above diagrams.



This is exactly the problem of a charged sphere with $\sigma(\theta)$ specified on the surface.

This was solved using the general solution to Laplace's Equation in spherical coordinates, written as a sum over Legendre polynomials. Example 3.9 (pages 144-146) shows

$$V(r, \theta) = \begin{cases} \frac{P}{3\epsilon_0} r \cos \theta & r \leq R \quad \text{inside the polarized sphere} \\ \frac{P}{3\epsilon_0} \frac{R^3}{r^2} \cos \theta & r \geq R \quad \text{outside the polarized sphere} \end{cases}$$

The potential inside is r -dependent because the surface of the sphere is not uniformly charged. (otherwise it would be r -independent)

The electric field is given by $\vec{E} = -\vec{\nabla}V$, where we use the gradient expression in Cartesian coordinates for $r \leq R$ (since $r \cos \theta = z$), and spherical coordinates for the $r \geq R$ case,

$$\vec{E} = \begin{cases} -\frac{1}{3\epsilon_0} \vec{P} & r < R \\ \frac{\vec{P}}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) & r \geq R \end{cases} \quad \text{** This is a key result that will be used later **}$$

with $\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$ the total dipole moment of the entire sphere.

Inside the sphere the electric field is uniform and in the $-\hat{z}$ direction. This is called the “de-polarization” field. Outside the electric field is simply that of a dipole at the center of the sphere with a total dipole moment equal to that of the entire sphere. Note that the dipole field is only present outside of the polarized sphere where $r > R$.



Note that the $\vec{E}$ -field lines DO NOT “circulate” in/around this object!

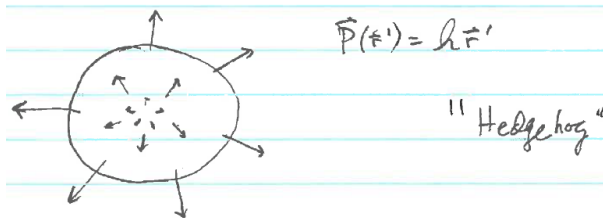
Example: Problem 4.10

Consider a sphere of radius R with polarization

$$\vec{P} = k\vec{r}$$

when k is a constant.

- What is the bound charge?
- What is the electric field inside and outside the sphere.



- Clearly there is divergence of $\vec{P}$

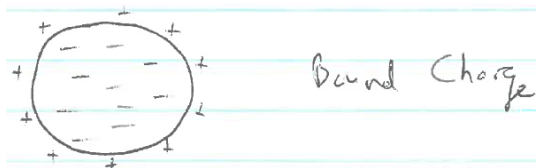
$$\begin{aligned}\rho_b &= -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 k r) + 0 + 0 \\ &= -\frac{k}{r^2} 3r^2 = -3k\end{aligned}$$

The bound charge per unit volume is uniform and negative.

The surface charge density is also uniform

$$\sigma_b = \vec{P} \cdot \hat{n} = k\vec{r} \cdot \hat{n} = k\vec{r} \cdot \hat{r} \Big|_R = kR$$

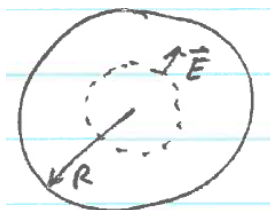
This is a positive charge per unit area.



Now, remarkably, we can replace the original polarized-sphere problem with a much simpler electrostatics problem involving just these bound charges. This new problem is very similar to those that we have solved before.

b) Find the electric field everywhere

Inside the sphere there is a uniform charge density. We expect a spherically-symmetric electric field that only depends on the radial coordinate r . Use Gauss's law to find the electric field,



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$E(r) 4\pi r^2 = \rho_b \frac{4}{3}\pi r^3 / \epsilon_0 = -3k \frac{4}{3}\pi r^3 / \epsilon_0$$

so

$$E(r) = \frac{-kr}{\epsilon_0}$$

or

$$\vec{E} = -\frac{k\vec{r}}{\epsilon_0} = \frac{-\vec{P}}{\epsilon_0} \quad \text{inside the sphere}$$

The electric field points towards the center of the sphere. This again is what we will later call the “de-polarization field”.

Outside the sphere $r > R$ we again expect spherical symmetry for the electric field. We can also use Gauss’s Law,



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

What is the charge enclosed in the Gaussian surface, Q_{enc} ? $Q_{enc} = \frac{4}{3}\pi R^3 \rho_b + 4\pi R^2 \sigma_b$

$$\begin{aligned} Q_{enc} &= \frac{4}{3}\pi R^3 (-3k) + 4\pi R^2 kR \\ &= -4\pi R^3 k + 4\pi R^3 k \\ &= 0 \end{aligned}$$

So $\vec{E} = 0$ outside this sphere since it has zero net charge and the charge is distributed with spherical symmetry. By the way, the result that the total volume bound charge plus the total surface bound charge is zero, calculated above, is not an accident. If an object has zero monopole moment ($Q = 0$), then when it becomes polarized by stretching atoms or orienting molecules, it cannot magically create a net charge. This means that total bound charge must be zero, quite generally. You prove this for homework in problem 4.14.