

Lecture #17 Summary

PHYS 313 Electricity and Magnetism I

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We are now going to look at how arbitrary charge distributions can be described in terms of the potential, and the electric field, as a function of distance from the center of such distributions. This leads to the concept of a **multi-pole expansion** for V and $\vec{E}$.

Multipole Expansion

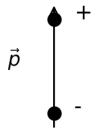
A charge distribution can be thought of as a linear combination of “ 2^n -poles,” as

$$\rho(\vec{r}') = c_1 \cdot (\text{monopole}) + c_2 \cdot (\text{dipole}) + c_3 \cdot (\text{quadrupole}) + c_4 \cdot (\text{octopole}) + \dots$$

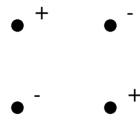
monopole



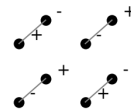
dipole



quadrupole



octopole



This is a kind of basis set for describing an arbitrary charge distribution. This is also the starting point for graduate-level electrostatics.

Consider the potential $V(\vec{r})$ due to an arbitrary three-dimensional charge distribution:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{Vol}} \frac{\rho(\vec{r}') d\tau'}{z}$$

Looking at this triangle between $\vec{r}$, $\vec{r}'$ and $\vec{z}$, and using the law of cosines:

$$z^2 = r^2 + (r')^2 - 2rr' \cos \alpha$$

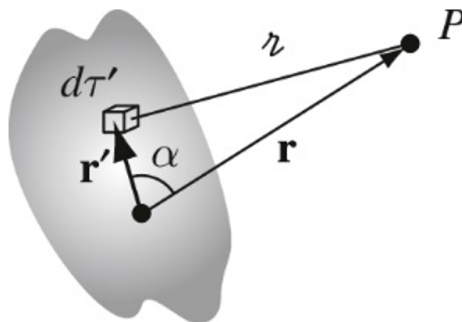


Figure 1: Definition of vectors $\vec{r}$, $\vec{r}'$, and $\vec{z}$ and the angle α in an integral over a charge density ρ .

$$= r^2 \left[1 + \left(\frac{r'}{r} \right)^2 - 2 \frac{r'}{r} \cos \alpha \right]$$

or

$$z = r\sqrt{1 + \epsilon}$$

with

$$\epsilon \equiv \left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos \alpha = \left(\frac{r'}{r} \right) \left[\frac{r'}{r} - 2 \cos \alpha \right]$$

If you are far from the charge distribution, then $r \gg r'$, so $\epsilon \ll 1$. In this case we can do a binomial expansion for $1/z$ and the potential:

$$\text{Recall the binomial expansion} \quad (1 + \epsilon)^n = 1 + n\epsilon + \frac{n(n-1)}{2!}\epsilon^2 + \frac{n(n-1)(n-2)}{3!}\epsilon^3 + \dots$$

$$\text{so } \frac{1}{z} = \frac{1}{r\sqrt{1 + \epsilon}} = \frac{1}{r}(1 + \epsilon)^{-1/2}$$

$$\approx \frac{1}{r} \left[1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots \right]$$

Putting back r , r' , and α :

$$\frac{1}{z} = \frac{1}{r} \left\{ 1 - \frac{1}{2} \left(\frac{r'}{r} \right) \left(\frac{r'}{r} - 2 \cos \alpha \right) + \frac{3}{8} \left(\frac{r'}{r} \right)^2 \left(\frac{r'}{r} - 2 \cos \alpha \right)^2 - \frac{5}{16} \left(\frac{r'}{r} \right)^3 \left(\frac{r'}{r} - 2 \cos \alpha \right)^3 + \dots \right\}$$

Combining like-powers of (r'/r) :

$$\frac{1}{z} = \frac{1}{r} \left\{ 1 + \frac{r'}{r} \cos \alpha + \left(\frac{r'}{r} \right)^2 \frac{1}{2} (3 \cos^2 \alpha - 1) + \left(\frac{r'}{r} \right)^3 \frac{1}{2} (5 \cos^3 \alpha - 3 \cos \alpha) + \dots \right\}$$

But look at these terms:

- The first is $P_0(\cos \alpha)$.
- The second contains $P_1(\cos \alpha)$.
- The third contains $P_2(\cos \alpha) = (3 \cos^2 \alpha - 1)/2$.
- The fourth contains $P_3(\cos \alpha) = (5 \cos^3 \alpha - 3 \cos \alpha)/2$.

A trend is developing here:

$$\boxed{\frac{1}{z} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r} \right)^{\ell} P_{\ell}(\cos \alpha)} \quad \left(\frac{r'}{r} \ll 1 \text{ assumed} \right)$$

where α is the angle between $\vec{r}'$ and $\vec{r}$ as shown in Figure 1 above.

Now put this into the general expression for the potential:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int_{\text{Vol}} \rho(\vec{r}') (r')^{\ell} P_{\ell}(\cos \alpha) d\tau'}$$

(NOTE that α is *not* the angle between the z -axis and the source vector $\vec{r}'$.)

This is the multipole expansion for the potential. Each term in the sum is a “pole” of the charge distribution:

$$\ell = 0 \text{ Monopole} \quad \ell = 1 \text{ Dipole} \quad \ell = 2 \text{ Quadrupole} \quad \ell = 3 \text{ Octopole}$$

Aside: General Expression (Graduate E&M)

A more general expression, used in graduate E&M (Jackson Electrodynamics, Eq. 3.38), is

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} P_{\ell}(\cos \alpha)$$

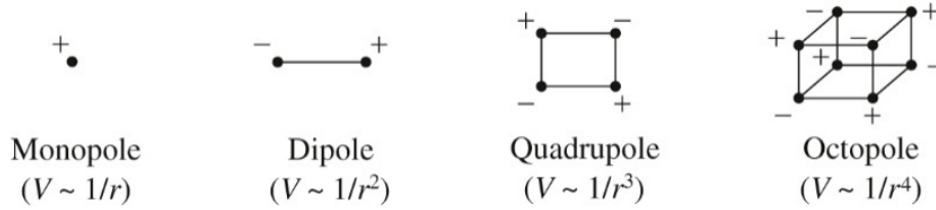
where $r_{<}$ and $r_{>}$ denote the lesser and greater of $|\vec{r}|$ and $|\vec{r}'|$. Here α is the angle between $\vec{r}$ and $\vec{r}'$.

Back to our multi-pole expansion...

Look at the multipole expansion for the potential term by term:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{1}{r} \int \rho(\vec{r}') d\tau' + \frac{1}{r^2} \int r' \cos \alpha \rho(\vec{r}') d\tau' \right. \\ \left. + \frac{1}{r^3} \int (r')^2 \frac{1}{2} (3 \cos^2 \alpha - 1) \rho(\vec{r}') d\tau' \right. \\ \left. + \frac{1}{r^4} \int (r')^3 \frac{1}{2} (5 \cos^3 \alpha - 3 \cos \alpha) \rho(\vec{r}') d\tau' + \dots \right\}$$

Note that each term in the expansion falls off with a higher power of r . This expansion is a good way to calculate and estimate the potential at large distances (large compared to the size of the charge distribution).



1 The Monopole Term

The first term in the multipole expansion is the monopole:

$$Q = \int \rho(\vec{r}') d\tau'$$

So

$$V_{\text{monopole}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

Q is called the **monopole moment** of the charge distribution. This is the dominant term in the potential. If $Q = 0$, then the next term dominates.

2 Dipole Term

$$\vec{p} \equiv \int \vec{r}' \rho(\vec{r}') d\tau'$$

defines the **dipole moment** of a charge distribution. Note that $\vec{p}$ can depend on the origin of the coordinate system used to calculate the integral!

So the dipole term of the potential expansion is

$$V_{\text{dipole}}(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

where

$$\vec{p} \cdot \hat{r} = \int r' \cos \alpha \rho(\vec{r}') d\tau',$$

since $r' \cos \alpha = \hat{r} \cdot \vec{r}'$ (see Figure 1 above). The dipole moment clearly depends on the geometry of the charge distribution (and the choice of origin if $Q \neq 0$).

3 Quadrupole Term

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int (r')^2 \frac{1}{2} (3 \cos^2 \alpha - 1) \rho(\vec{r}') d\tau'$$

The quadrupole term can be written in terms of the quadrupole moment tensor. Jackson (p. 146) writes

$$V_{\text{quad}} = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{i,j=1}^3 \frac{x_i x_j}{r^5} Q_{ij}$$

$$V_{\text{quad}}(r) = \frac{1}{4\pi\epsilon_0} \frac{1}{2r^3} \sum_{i,j=1}^3 \hat{x}_i \hat{x}_j Q_{ij}$$

(Note: $\hat{x}_i = (\hat{x}, \hat{y}, \hat{z})$ for $i = 1, 2, 3$.)

$$Q_{ij} \equiv \int [3x'_i x'_j - (r')^2 \delta_{ij}] \rho(\vec{r}') d\tau'$$

The ℓ^{th} multipole coefficient has $2\ell + 1$ pieces:

- $\ell = 0$: 1 (Q)
- $\ell = 1$: 3 (p_x, p_y, p_z)
- $\ell = 2$: 5 ($\text{Tr } \bar{\bar{Q}} = 0 \Rightarrow 6 - 1$ terms)

$$V_{\text{quad}} \sim \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

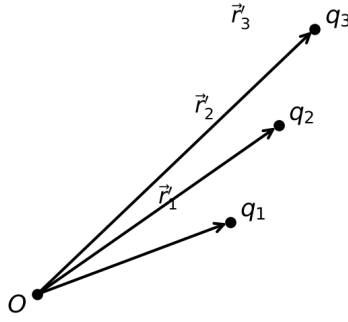
4 The Dipole Term

The dipole moment

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau'$$

can be written as follows for point charges:

$$\vec{p} = \sum_{i=1}^N q_i \vec{r}'_i$$



Note that the dipole moment vector depends on your choice of origin for the coordinate system. Only when $Q = 0$ is the dipole moment independent of origin. Here is a brief proof:

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau'$$

Now displace the origin by $\vec{a}$; the new dipole moment is

$$\vec{p}' = \int (\vec{r}' - \vec{a}) \rho(\vec{r}') d\tau'$$

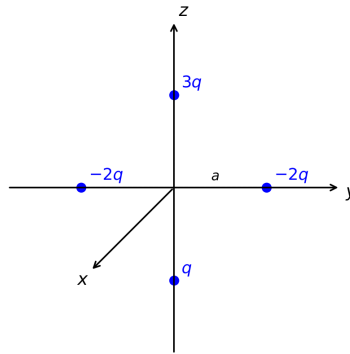
$$= \vec{p} - \vec{a} \int \rho(\vec{r}') d\tau'$$

$$= \vec{p} - \vec{a} Q$$

$$\boxed{\vec{p}' = \vec{p}} \quad \text{since } Q = 0 \text{ by assumption.}$$

5 Example of Multipole Expansion (Problem 3.31)

Consider four charges, as shown below, all at a distance a from the origin.



We want to find an approximate expression for the potential valid for $r \gg a$. We will use the multipole expansion.

5.1 The Monopole Term

$$Q = 3q - 2q + q - 2q = 0$$

The monopole moment Q is zero, so there is no monopole term in the expansion.

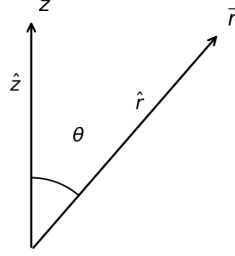
5.2 Dipole Term

$$\vec{p} = \sum_{i=1}^4 q_i \vec{r}_i' = 3qa \hat{z} - 2qa \hat{y} - qa \hat{z} + 2qa \hat{y}$$

$$\vec{p} = 2qa \hat{z}$$

The leading term in the potential far away ($r \gg a$) is:

$$V(\vec{r}) \approx V_{\text{dipole}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{2qa \hat{z} \cdot \hat{r}}{r^2}$$



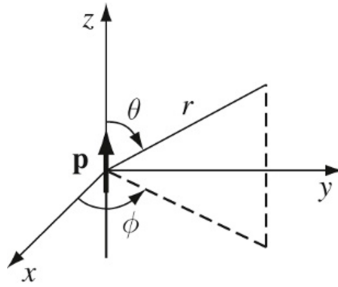
We can write the potential in spherical coordinates now since the dipole is oriented along the $\hat{z}$ -direction:

$$V_{\text{dipole}}(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{2qa \cos \theta}{r^2}$$

This charge distribution is *not* azimuthally symmetric. Since $\vec{p} \sim \hat{z}$, it preserves azimuthal symmetry for $\ell = 1$. To capture the ϕ -dependence, the $\ell = 2$ term, and higher, must be used!

6 Electric Field of a Dipole

Make the dipole moment point in the z -direction and lie at the origin. With this we can express the potential and electric field in spherical coordinates.

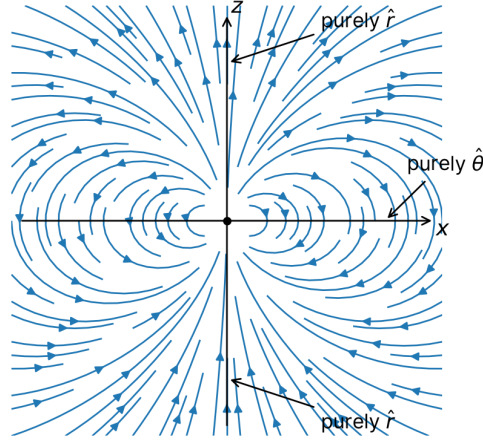


$$V_{\text{dipole}}(r, \theta) = \frac{\hat{r} \cdot \vec{p}}{4\pi\epsilon_0 r^2} = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}, \quad \vec{p} = p \hat{z}$$

The electric field is given by

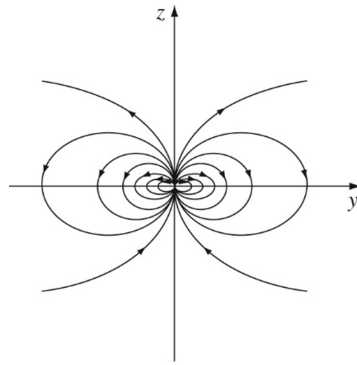
$$\begin{aligned} \vec{E}_{\text{dipole}} &= -\vec{\nabla} V = -\frac{\partial V}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + 0 \hat{\phi} \\ &= \frac{2p \cos \theta}{4\pi\epsilon_0 r^3} \hat{r} + \frac{p \sin \theta}{4\pi\epsilon_0 r^3} \hat{\theta} + 0 \hat{\phi} \end{aligned}$$

$$\vec{E}_{\text{dipole}} = \frac{p}{4\pi\epsilon_0 r^3} \left(2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right)$$

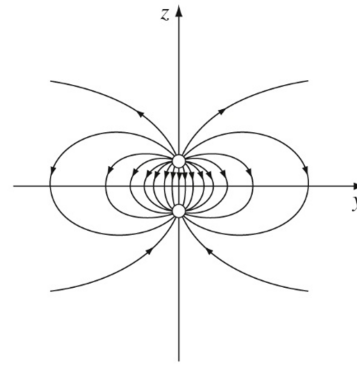


All the field lines start and end on the charges making up the dipole.

Note that Griffiths makes a distinction between a “physical dipole” and a “pure dipole”. A physical dipole is made up of two charges, $+q$ and $-q$ separated by a finite-length displacement vector $\vec{d}$, creating a dipole with moment $\vec{p} = q\vec{d}$. This object has large fields between the charges, as shown below. The pure dipole is an idealized version in which the displacement vector $\vec{d}$ is sent to zero length, and the charge q diverges, such that $\vec{p}$ remains finite. This pure dipole actually hides a double-delta function, see below!



(a) Field of a “pure” dipole



(b) Field of a “physical” dipole

7 The Dipole Field and the ‘Hidden’ Double Delta Function

The field produced by an electric dipole is given by,

$$\vec{E}_{\text{dipole}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3} \left(2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right)$$

Just wondering: Let’s check what charge distribution gave rise to this electric field using $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

$$\begin{aligned} \vec{\nabla} \cdot \vec{E}_{\text{dipole}} &= \frac{p}{4\pi\epsilon_0} \left\{ \frac{2 \cos \theta}{r^2} \frac{\partial}{\partial r} \left(\frac{1}{r} \right) + \frac{1}{r^3} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin^{-1} \theta) \right\} \\ &= \frac{p}{4\pi\epsilon_0} \left\{ \frac{-2 \cos \theta}{r^4} + \frac{2 \sin \theta \cos \theta}{r^4 \sin \theta} \right\} \\ &= 0 \end{aligned}$$

But this cannot be true! $\vec{\nabla} \cdot \vec{E} = +\rho/\epsilon_0$ and there is a $+$ and $-$ charge making up the dipole!

The calculation of $\vec{E}_{\text{dipole}}$ from V_{dipole} missed a delta function:

$$\vec{E}_{\text{dipole}}(\vec{r}, \theta) = \frac{p}{4\pi\epsilon_0 r^3} \left(2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) - \frac{\vec{p}}{3\epsilon_0} \delta^3(\vec{r})$$

$$\vec{\nabla} \cdot \vec{E}_{\text{extra}} = \vec{\nabla} \cdot \left(\frac{\delta^3(\vec{r})}{3\epsilon_0} \vec{p} \right) = \frac{\delta^3(\vec{r})}{3\epsilon_0} \vec{\nabla} \cdot \vec{p} + \vec{p} \cdot \vec{\nabla} \left(\frac{\delta^3(\vec{r})}{3\epsilon_0} \right)$$

(the first term is 0 since $\vec{p} = \text{const.}$)

$$\rho_{\vec{p}} = \epsilon_0 \vec{\nabla} \cdot \vec{E}_{\text{extra}} = -\frac{1}{3} \vec{p} \cdot \vec{\nabla} \delta^3(\vec{r})$$

The figure below illustrates that the gradient of the delta function creates the two charges making up a “pure” dipole, as defined in Griffiths.

