

# Lecture #16 Summary

## PHYS 313 Electricity and Magnetism I

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### Another Way to Solve Laplace's Equation $\nabla^2 V = 0$

#### Separation of Variables

In the last lecture we did separation of variables in Cartesian coordinates following Griffiths, section 3.3.1.

I am going to do separation of variables in spherical coordinates since this turns out to be the most useful approach that will pay dividends in the future.

We are solving  $\nabla^2 V = 0$  in a region bounded by a surface where the potential  $V$  or the charge density  $\sigma$  is specified.

The basic idea of separation of variables is to write out the full  $V$  function in terms of a product of functions, each for one variable:

$$\text{Cartesian: } V(x, y, z) = X(x) Y(y) Z(z)$$

$$\text{Cylindrical: } V(s, \phi, z) = S(s) \Phi(\phi) Z(z)$$

$$\text{Spherical: } V(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

Substitute this ansatz into  $\nabla^2 V = 0$  and “separate variables” to produce 3 ordinary differential equations from one partial differential equation.

However, there is a price to pay for assuming such an oversimplified form of the solution. That is, we generally find an *infinite* number of such solutions, and we need to include all of them to construct the correct solution that satisfies our boundary conditions. In addition, there can exist solutions to Laplace's equations that cannot be expressed in this form, and we miss these solutions altogether.

#### Spherical Coordinates

We want to solve  $\nabla^2 V = 0$  for objects with spherical boundary conditions. As an example, let's consider a sphere of radius  $R$  with a potential that depends on  $\theta$  on its surface.

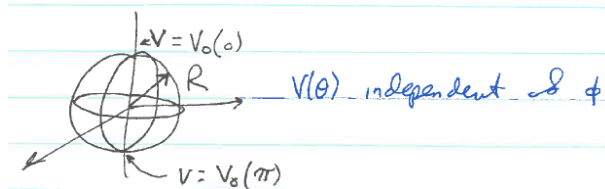


Figure 1: \*

Sphere of radius  $R$  with surface potential  $V = V_0(\theta)$ , independent of  $\phi$ .

Our objective is to solve for  $V(r, \theta, \phi)$  inside and outside this sphere, subject to the boundary condition  $V(R, \theta, \phi) = V_0(\theta)$ , where  $V_0$  is assumed independent of azimuthal angle  $\phi$ .

Start by writing out  $\nabla^2 V(r, \theta, \phi) = 0$  in spherical coordinates, from the inside front cover of the textbook:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Now, to simplify the problem, we shall assume azimuthal symmetry, as stated above. The problem is the same for all rotations about the  $z$ -axis, i.e. it is independent of the  $\phi$  coordinate. This allows us to drop the last term (since  $\partial V / \partial \phi = 0$ ), and after multiplying through by  $r^2$  we have:

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

## Separation of Variables in Spherical Coordinates

Now separate variables with the ansatz  $V(r, \theta) = R(r) \Theta(\theta)$ :

$$\frac{\partial}{\partial r} \left( r^2 \frac{dR}{dr} \Theta \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta R \frac{d\Theta}{d\theta} \right) = 0$$

Divide through by the product  $R(r) \Theta(\theta)$ :

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

Look at this equation carefully. It says that

$$f(r) + g(\theta) = 0$$

for all  $r$  and  $\theta$ . How can this be?  $r$  and  $\theta$  are two independent orthogonal components, and this sum can equal zero only under one very special circumstance. Both  $f(r)$  and  $g(\theta)$  must be constant:

$$f(r) = C, \quad g(\theta) = -C \quad \implies \quad f(r) + g(\theta) = C - C = 0$$

So

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = C \quad \text{and} \quad \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = -C$$

It turns out that the constant  $C$  can be written as  $C = \ell(\ell + 1)$ . [This makes the  $R(r)$  solution particularly simple.] Note that  $\ell$  is not yet determined.

## The Radial Equation

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \ell(\ell + 1) R$$

or equivalently,

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \ell(\ell + 1) R = 0$$

This ordinary differential equation has two simple solutions:

$$\boxed{R(r) = A r^\ell + \frac{B}{r^{\ell+1}}}$$

Note:  $A r^\ell$  is regular as  $r \rightarrow 0$  but diverges as  $r \rightarrow \infty$ .  
 $B / r^{\ell+1}$  is regular as  $r \rightarrow \infty$  but diverges as  $r \rightarrow 0$ .

## Angular Equation — Legendre's Equation

Now look at the angular ( $\theta$ ) equation:

$$\frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \ell(\ell+1) \sin \theta \Theta = 0$$

This is called Legendre's equation. It comes up in the solution of the Schrödinger Equation for the Hydrogen Atom, where the positively-charged nucleus has a Coulomb attraction to the negatively-charged electron:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi - \frac{e^2}{4\pi\epsilon_0 r} \psi = E\psi$$

Make a substitution change of variable:

$$u = \cos \theta \quad \implies \quad \sin \theta = \sqrt{1 - u^2} = \sqrt{1 - \cos^2 \theta}$$

$\theta = 0 \rightarrow \pi$  now corresponds to  $u = 1 \rightarrow -1$ . In other words the substitution  $u = \cos \theta$  maps  $\theta \in [0, \pi]$  to  $u \in [-1, 1]$ .

Also change the function from  $\Theta(\theta)$  to  $P(u)$ :

$$\Theta(\theta) = P(u)$$

So

$$\frac{d\Theta}{d\theta} = \frac{dP}{du} \frac{du}{d\theta} = -\sin \theta \frac{dP}{du} = -\sqrt{1 - u^2} \frac{dP}{du}$$

This transforms Legendre's Equation to

$$-\sqrt{1 - u^2} \frac{d}{du} \left( -\sqrt{1 - u^2} \sqrt{1 - u^2} \frac{dP}{du} \right) + \ell(\ell+1) \sqrt{1 - u^2} P = 0$$

or

$$\boxed{\frac{d}{du} \left( (1 - u^2) \frac{dP}{du} \right) + \ell(\ell+1) P = 0, \quad u \in [-1, 1]}$$

We demand solutions for  $P(u)$  that remain finite for  $u$  between  $-1$  and  $1$ , inclusive.

The fascinating details of this equation is solved are reviewed in the Appendix 1 at the end of this document.

## Legendre Polynomials

The solutions to the differential equation are finite polynomials of degree  $\ell$ . They are called the **Legendre Polynomials**:

$$\boxed{\begin{array}{ll} n_{\max} = 0 = \ell : & P_0(u) = 1 \\ n_{\max} = 1 = \ell : & P_1(u) = u \\ n_{\max} = 2 : & P_2(u) = (3u^2 - 1)/2 \\ n_{\max} = 3 : & P_3(u) = (5u^3 - 3u)/2 \\ & P_4(u) = (35u^4 - 30u^2 + 3)/8 \end{array}}$$

They are normalized so that  $P_\ell(1) = 1$ .

The Legendre functions constitute a complete orthonormal set on the interval  $[-1, 1]$ .

Note that the parity of  $P_\ell(u)$  is  $(-1)^\ell$ :

$$\begin{array}{ll} \ell \text{ even} : & P_\ell(-u) = P_\ell(u) \\ \ell \text{ odd} : & P_\ell(-u) = -P_\ell(u) \end{array}$$

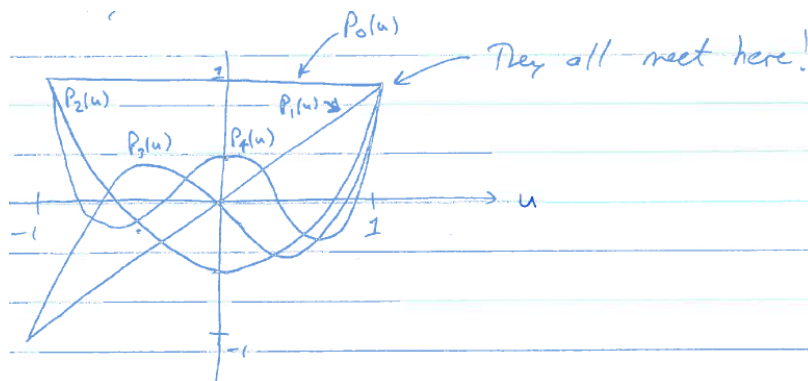


Figure 2: \*  
Legendre polynomials  $P_0$  through  $P_4$  on the interval  $[-1, 1]$ . They all meet at  $u = 1$ .

The Legendre Polynomials can be generated by the **Rodrigues Formula**:

$$P_\ell(u) = \frac{1}{2^\ell \ell!} \left( \frac{d}{du} \right)^\ell (u^2 - 1)^\ell$$

A choice of  $\ell$  gives you either an even or odd solution, depending on whether  $\ell$  is even or odd.

The other set of solutions to the second order differential equation are called the **Legendre Function of the Second Kind**  $Q_\ell(u)$ . See E.L. Ince, *ODE*, p. 165.

When  $\ell$  is not an integer, there is an infinite series solution that diverge at  $u = \pm 1$  (which means  $\theta = 0$  or  $\pi$ ), in general, so are not useful for physical problems.

## Properties of Legendre Functions

The Legendre functions form a complete orthogonal set of functions on the interval  $[-1, 1]$ .

**Completeness:** Any well-behaved function  $f(u)$  for  $-1 \leq u \leq 1$  can be expressed as a linear combination of Legendre functions.

$$f(u) = \sum_{m=0}^{\infty} c_m P_m(u) \quad -1 \leq u \leq 1$$

**Orthogonality:** The integral of a product of any two of the functions is zero unless the two functions are the same:

$$\int_{-1}^1 P_\ell(u) P_{\ell'}(u) du = \begin{cases} 0 & \ell \neq \ell' \\ \frac{2}{2\ell + 1} & \ell' = \ell \end{cases}$$

These are very useful properties of the Legendre functions.

Now translate all of these results back to  $\theta$ : with  $u = \cos \theta$ ,  $du = -\sin \theta d\theta$ ,

$$\begin{aligned} - \int_{\pi}^0 P_\ell(\cos \theta) P_{\ell'}(\cos \theta) \sin \theta d\theta &= \delta_{\ell\ell'} \frac{2}{2\ell + 1} \\ \int_0^{\pi} P_\ell(\cos \theta) P_{\ell'}(\cos \theta) \sin \theta d\theta &= \delta_{\ell\ell'} \frac{2}{2\ell + 1} \end{aligned} \tag{1}$$

Recall the Kronecker delta:  $\delta_{\ell\ell'} = 1$  if  $\ell = \ell'$  and 0 if  $\ell \neq \ell'$ .

## Solution of Laplace's Equation

Now return to the solution of Laplace's equation, using our new-found result.

$$\Theta(\theta) = P(u) = P_\ell(\cos \theta)$$

Recall that

$$V(r, \theta) = R(r) \Theta(\theta) = \left( Ar^\ell + \frac{B}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

where  $\ell$  is an integer,  $\ell \in (0, 1, 2, 3, \dots)$ .

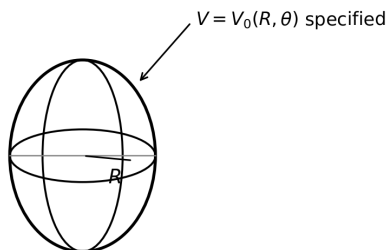
However, in order to satisfy an arbitrary boundary condition in  $(r, \theta)$  we must include all possible answers to the equation:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left( A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta) \quad (2)$$

This is the most general solution for  $\nabla^2 V = 0$  with  $V$  independent of  $\phi$  in spherical coordinates. To proceed, we need to enforce some boundary conditions.

## Example Problem

The potential  $V_0(\theta)$  is specified on the surface of a hollow sphere of radius  $R$ . Find the potential inside the sphere.



The general solution is

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left( A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

How do we determine the infinite set of coefficients  $A_0, A_1, A_2, \dots$  and  $B_0, B_1, B_2, \dots$ ? Enforce the boundary conditions:

- (1)  $V(r)$  remains finite everywhere.
- (2)  $V(r \rightarrow \infty)$  goes to zero.
- (3)  $V(R, \theta) = V_0(\theta)$ , the specified potential distribution.

Enforce boundary condition (1)  $\Rightarrow$  the  $B_\ell/r^{\ell+1}$  terms diverge as  $r \rightarrow 0$  for all  $\ell = 0, 1, 2, \dots$ . So set  $B_\ell = 0$  for all  $\ell$ .

Boundary condition (2) does not apply here directly because we only need the solution out to  $r = R$ .

Now enforce boundary condition (3):

$$V_0(\theta) = V(R, \theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos \theta)$$

Since  $P_\ell(\cos \theta)$  form a complete set, we can express  $V_0(\theta)$  as an infinite series, exactly as we have in the above equation.

How do we determine the  $A_\ell$ ? Use the orthogonality property (aka the “sum buster” technique).

Multiply both sides by  $P_{\ell'}(\cos \theta) \sin \theta$ , where  $\ell'$  is an arbitrary positive integer or zero, and integrate on  $\theta$  from 0 to  $\pi$ :

$$\int_0^\pi V_0(\theta) P_{\ell'}(\cos \theta) \sin \theta d\theta = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi P_\ell(\cos \theta) P_{\ell'}(\cos \theta) \sin \theta d\theta$$

The integral on the RHS is just the Legendre polynomial orthogonality condition, Eq. 1, above.

$$= \sum_{\ell=0}^{\infty} A_{\ell} R^{\ell} \delta_{\ell\ell'} \frac{2}{2\ell+1} = \frac{2A_{\ell'} R^{\ell'}}{2\ell'+1}$$

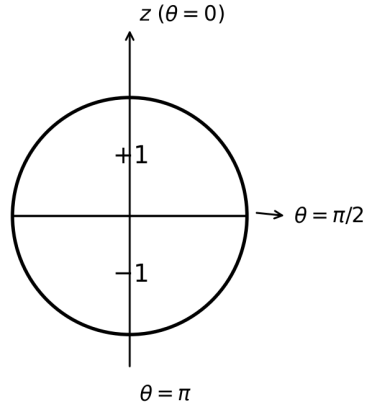
Solving for  $A_{\ell}$ , after renaming the dummy variable, we have:

$$A_{\ell} = \left( \frac{2\ell+1}{2} \right) \frac{1}{R^{\ell}} \int_0^{\pi} V_0(\theta) P_{\ell}(\cos \theta) \sin \theta d\theta$$

## North/South Hemisphere Example

To proceed, suppose now that the potential is maintained at  $V_0 = +1$  on the northern hemisphere and  $V_0 = -1$  on the southern hemisphere (and there is a very small gap between the conductors at the equator):

$$V_0(\theta) = \begin{cases} 1 & 0 \leq \theta < \pi/2 \\ -1 & \pi/2 < \theta \leq \pi \end{cases}$$



What are the coefficients  $A_{\ell}$ ?

$$A_{\ell} = \left( \frac{2\ell+1}{2} \right) \frac{1}{R^{\ell}} \int_0^{\pi} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} P_{\ell}(\cos \theta) \sin \theta d\theta$$

where  $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$  is the  $V_0$  function and it is an odd function of  $\theta$  about  $\pi/2$ , and  $P_{\ell}(\cos \theta)$  has parity  $(-1)^{\ell}$  about  $\theta = \pi/2$ . Note that  $\sin \theta$  is even about  $\theta = \pi/2$ . These symmetries allow us to make the following conclusions:

- If  $\ell$  is even: the product of 1 odd and 1 even function is odd, making the integral zero. (consider for example  $P_0(x) = 1$ .)
- If  $\ell$  is odd: the product of 2 odd functions is an even function, making the integral non-zero. (consider for example  $P_1(x) = x$ .)

$\Rightarrow A_{\ell} = 0$  for  $\ell$  even.

For  $\ell$  odd, we have

$$A_{\ell} = \frac{2\ell+1}{2} \frac{1}{R^{\ell}} 2 \int_0^{\pi/2} P_{\ell}(\cos \theta) \sin \theta d\theta = \frac{2\ell+1}{R^{\ell}} \frac{P'_{\ell}(0)}{\ell(\ell+1)}$$

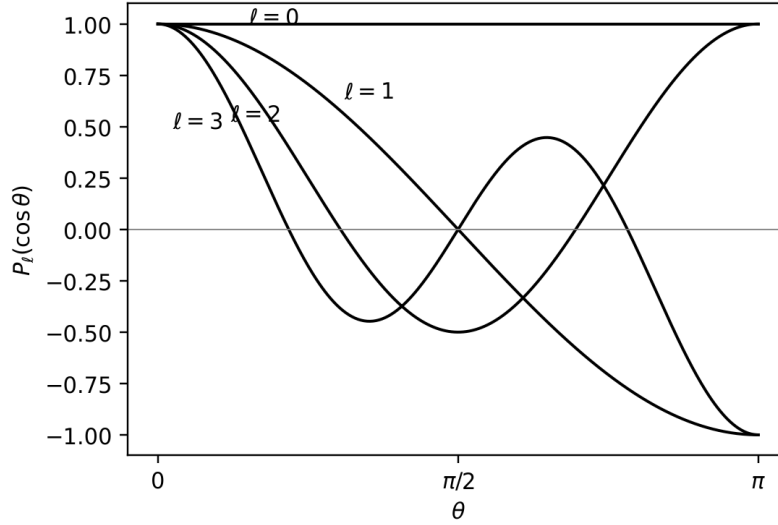
so

$$A_{\ell} = \frac{2\ell+1}{\ell(\ell+1)} \frac{1}{R^{\ell}} P'_{\ell}(0) \quad \text{where } \ell \text{ is odd,}$$

where  $P'_\ell(0)$  is the derivative of  $P_\ell(u)$  evaluated at  $u = 0$ :

$$P'_\ell(0) = \left. \frac{dP_\ell(u)}{du} \right|_{u \rightarrow 0}$$

Below is a plot of  $P_\ell(\cos \theta)$  for  $\ell = 0, 1, 2, 3$ :



The first few expansion coefficients are:

$$A_1 = \frac{3}{2} \frac{1}{R} \quad A_3 = -\frac{7}{8} \frac{1}{R^3} \quad A_5 = \frac{11}{16} \frac{1}{R^5}$$

**Final Result**

$$V(r, \theta) = \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \frac{1}{R^\ell} P'_\ell(0) r^\ell P_\ell(\cos \theta) \quad \text{Holds for } r \leq R$$

This satisfies the boundary conditions at  $r = 0$ ,  $r = R$ .

The case  $r > R$  is similar. Just make  $A_\ell = 0$  and calculate the  $B_\ell$  coefficients:

$$V(r, \theta) = \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} P'_\ell(0) \left( \frac{R}{r} \right)^{\ell+1} P_\ell(\cos \theta)$$

Plots of the potential inside the sphere are shown on the class website.

## The Electric Field

What is the electric field arising from this potential?

$$\begin{aligned} \vec{E} &= -\vec{\nabla} V = -\frac{\partial V}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} \\ &= -\sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{2\ell+1}{\ell+1} \frac{1}{R^\ell} P'_\ell(0) r^{\ell-1} P_\ell(\cos \theta) \hat{r} + \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} \frac{1}{R^\ell} P'_\ell(0) r^{\ell-1} \sin \theta P'_\ell(\cos \theta) \hat{\theta} \end{aligned}$$

But  $P'_\ell(\cos \theta) = \ell P_{\ell-1}(\cos \theta) + \cos \theta P'_{\ell-1}(\cos \theta)$  (from a recursion relation for the Legendre polynomials), so

$$\vec{E} = \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{\infty} \frac{2\ell+1}{\ell+1} \frac{1}{R^\ell} P'_\ell(0) r^{\ell-1} \left\{ -P_\ell(\cos \theta) \hat{r} + \frac{1}{\ell} \sin \theta (P_{\ell-1}(\cos \theta) + \ell \cos \theta P'_{\ell-1}(\cos \theta)) \hat{\theta} \right\}$$

# 1 Appendix: Solution to the Angular Equation

$$\boxed{\frac{d}{du} \left( (1-u^2) \frac{dP}{du} \right) + \ell(\ell+1)P = 0, \quad u \in [-1, 1]}$$

We demand solutions for  $P(u)$  that remain finite for  $u = \cos \theta$  between  $-1$  and  $1$ , inclusive.

## Power Series Solution

How do we solve this equation? Try the power series solution:

$$P(u) = \sum_{n=0}^{\infty} C_n u^n = C_0 + C_1 u + C_2 u^2 + C_3 u^3 + \dots$$

Substitute this into the equation. Expanding  $(1-u^2) \frac{d^2 P}{du^2} - 2u \frac{dP}{du} + \ell(\ell+1)P = 0$ :

$$(1-u^2) \sum_{n=0}^{\infty} n(n-1) C_n u^{n-2} - 2 \sum_{n=0}^{\infty} n C_n u^n + \ell(\ell+1) \sum_{n=0}^{\infty} C_n u^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)C_{n+2} - n(n-1)C_n - 2n C_n + \ell(\ell+1)C_n] u^n = 0$$

Note that  $n(n-1) + 2n = n(n+1)$ , so

$$\sum_{n=0}^{\infty} [(n+2)(n+1)C_{n+2} - n(n+1)C_n + \ell(\ell+1)C_n] u^n = 0$$

This must hold for all  $u$ . The only way this convergent sum can be zero for all  $u$  is if all the coefficients are zero. This leads to a recursion relation:

$$(n+2)(n+1)C_{n+2} = [n(n+1) - \ell(\ell+1)] C_n$$

or

$$\boxed{\frac{C_{n+2}}{C_n} = \frac{n(n+1) - \ell(\ell+1)}{(n+1)(n+2)}}$$

So given  $C_0$ , this recursion relation determines  $C_2, C_4, C_6, C_8, \dots$

$$n=0: \quad \frac{C_2}{C_0} = \frac{0 - \ell(\ell+1)}{2} = -\frac{\ell(\ell+1)}{2}$$

Given  $C_1$ , this recursion relation determines  $C_3, C_5, C_7$ , etc.

$$n=1: \quad \frac{C_3}{C_1} = \frac{2 - \ell(\ell+1)}{6} = \frac{1}{3} - \frac{\ell(\ell+1)}{6}$$

So there are two independent constants,  $C_0$  and  $C_1$ , that multiply the two linearly independent solutions to the second order differential equation.

$C_0$  labels the solution with even powers of  $u$  and is therefore an even function.

$C_1$  labels the solution with odd powers of  $u$  and is therefore an odd function.

## Convergence and Termination

Again we demand that the solution be regular for all  $u$  in the interval  $[-1, 1]$ . For this to happen, the power series solution must converge for  $u = \pm 1$ . This will NOT converge at  $u = \pm 1$  if the series is infinite. Look at the terms as  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} \frac{C_{n+2}}{C_n} = \frac{n^2}{n^2} = 1$$

So the infinite series will look like

$$P(u) \approx \dots + u^{799} + u^{800} + u^{801} + \dots \Rightarrow \sum_{n=0}^{\infty} u^n = \frac{1}{1-u}, \quad \text{which diverges as } u \rightarrow 1.$$

Hence to get a finite  $P(u)$ , we need to **TERMINATE** the series. Terminate the series at  $n = n_{\max}$ , such that all subsequent terms are zero:

$$n = n_{\max} : \quad \frac{C_{n_{\max}+2}}{C_{n_{\max}}} = \frac{n_{\max}(n_{\max} + 1) - \ell(\ell + 1)}{(n_{\max} + 1)(n_{\max} + 2)} = 0$$

$$\implies n_{\max} = \ell \quad \text{does the trick!}$$

We need to make  $\ell$  a positive integer (or zero) to terminate the series and recover a finite result for  $P(u)$  and  $\Theta(\theta)$ . (The other solution  $\rightarrow -(\ell + 1)$ .)