

Lecture #14 Summary

PHYS 313 Electricity and Magnetism I

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Another Way to Solve Laplace's Equation $\nabla^2 V = 0$

Separation of Variables

We are solving $\nabla^2 V = 0$ in a region bounded by a surface where the potential V or the charge density σ is specified.

The basic idea of separation of variables is to write out the full V function in terms of a product of functions, each for one variable:

$$\text{Cartesian: } V(x, y, z) = X(x) Y(y) Z(z)$$

$$\text{Cylindrical: } V(s, \phi, z) = S(s) \Phi(\phi) Z(z)$$

$$\text{Spherical: } V(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

Substitute this ansatz into $\nabla^2 V = 0$ and “separate variables” to produce 3 ordinary differential equations from one partial differential equation.

However, there is a price to pay for assuming such an oversimplified form of the solution. That is, we generally find an *infinite* number of such solutions, and we need to include all of them to construct the correct solution that satisfies our boundary conditions. In addition, there can exist solutions to Laplace's equations that cannot be expressed in this form, and we miss these solutions altogether.

Here we will focus on solving Laplace's equation in Cartesian coordinates, Write

$$V(x, y, z) = X(x) Y(y) Z(z) \tag{1}$$

$\nabla^2 V = 0$ becomes

$$\frac{d^2 X}{dx^2} Y Z + X \frac{d^2 Y}{dy^2} Z + X Y \frac{d^2 Z}{dz^2} = 0$$

Divide through by $V = XYZ$:

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0$$
$$f(x) + g(y) + h(z) = 0$$

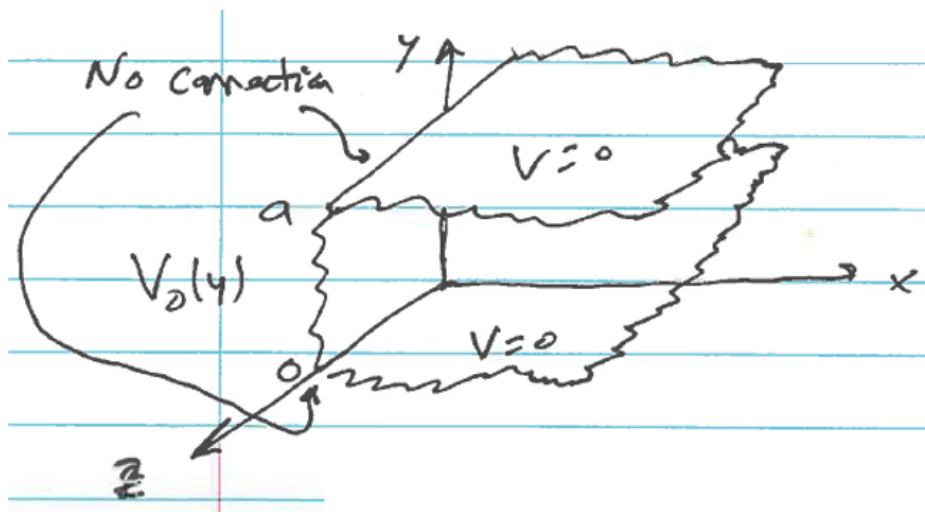
Suppose you vary the x location and $f(x)$ changes. The functions g and h have no dependence on x so they cannot vary to keep the sum equal to zero.

Hence it must be that each is constant!

$$f(x) = C_1, \quad g(y) = C_2, \quad h(z) = C_3 \quad \text{with } C_1 + C_2 + C_3 = 0$$

To proceed it helps to look at a particular problem to solve.

Consider the case in Griffiths Example 3.3



Boundary conditions:

- (i) $V = 0$ when $y = 0 \quad \forall x, z$
- (ii) $V = 0$ when $y = a \quad \forall x, z$
- (iii) $V = V_0(y)$ when $x = 0 \quad \forall z$. Here $V_0(y)$ will be specified later.
- (iv) $V \rightarrow 0$ as $x \rightarrow \infty$. Since only the segment at $x = 0$ has $V \neq 0$ here.

The problem is independent of z , so it reduces to a two-dimensional problem

$$V(x, y) = X(x) Y(y)$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

Now solve:

$$\frac{1}{X} \frac{d^2 X}{dx^2} = C_1 = k^2 > 0 \quad \text{This choice of a positive constant dictates the nature of the solution, as we shall see.}$$

$$\frac{d^2 X}{dx^2} = k^2 X \Rightarrow X(x) = Ae^{kx} + Be^{-kx}$$

It must be that

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -k^2 \quad \text{so that} \quad k^2 - k^2 = 0,$$

such that the sum of the two constants is zero!

$$\frac{d^2 Y(y)}{dy^2} = -k^2 Y \Rightarrow Y(y) = C \sin(ky) + D \cos(ky)$$

Hence

$$V(x, y) = (Ae^{kx} + Be^{-kx}) (C \sin(ky) + D \cos(ky))$$

What are the 4 constants?

$\Rightarrow$ Impose the boundary conditions.

* Condition (iv): $V \rightarrow 0$ as $x \rightarrow +\infty \Rightarrow A = 0$, otherwise V diverges.

So now

$$V(x, y) = e^{-kx} (C \sin(ky) + D \cos(ky))$$

(B has been combined with the other constants.)

* Condition (i): $V = 0$ at $y = 0 \Rightarrow D = 0$, since $\cos(k \cdot 0) = 1$.

So now

$$V(x, y) = Ce^{-kx} \sin(ky)$$

★ Condition (ii): $V = 0$ when $y = a$

$$0 = Ce^{-kx} \sin(ka)$$

There are an infinite number of solutions to this equation

$$ka = n\pi \quad n = 1, 2, 3, \dots$$

Note that $n < 0$ gives $k < 0$ and divergence in x . And $n = 0$ just gives $V = 0$ everywhere, which does not satisfy condition (iii), in general. So

$$k = \frac{n\pi}{a} \quad n = 1, 2, 3, \dots$$

You see now how choosing the sign of k^2 for the $X(x)$ equation is dictated by condition (iv).

⇒ In fact, no single value of n will allow us to satisfy condition (iii) $V = V_0(y)$ at $x = 0$. This is a crisis!

To get out of this jam, we can take a linear combination of *all* the n -solutions and use it to satisfy Boundary Condition (iii)!

This works because Laplace's equation is *linear*

$$\begin{aligned} V &= \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \dots \\ \nabla^2 V &= \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \alpha_3 \nabla^2 V_3 + \dots \\ &= \alpha_1 0 + \alpha_2 0 + \alpha_3 0 + \dots \\ &= 0 \quad \checkmark \end{aligned}$$

Now exploit this property to satisfy condition (iii):

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

General solution, but what are the C_n ?

(Note that this solution is no longer a simple function $X(x)Y(y)$, as shown in Eq. (1) above. It is in fact an infinite sum of such solutions!)

Condition (iii): $V(x = 0, y) = V_0(y)$ now demands:

$$V_0(y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right)$$

This is a Fourier sine series. Dirichlet's theorem says this can work for essentially any function $V_0(y)$!

So, how to find the coefficients C_n ?

Use the Fourier trick (better known as the "sum buster")

1. Multiply both sides by $\sin(n'\pi y/a)$, where n' is an arbitrary positive integer,
2. Integrate both sides from $y = 0$ to $y = a$:

$$\int_0^a V_0(y) \sin(n'\pi y/a) dy = \sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n'\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy$$

The integral on the RHS has two possible outcomes.

$$\int_0^a \sin\left(\frac{n'\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right) dy = \begin{cases} 0 & \text{when } n \neq n' \\ a/2 & \text{when } n = n' \end{cases}$$

It busts the sum!

$$\text{RHS} = \sum_{n=1}^{\infty} C_n \frac{a}{2} \delta_{nn'} \quad \delta_{nn'} = \begin{cases} 1 & n = n' \\ 0 & n \neq n' \end{cases} \quad \text{the Kronecker delta}$$

Hence

$$RHS = \frac{a}{2} C_{n'}$$

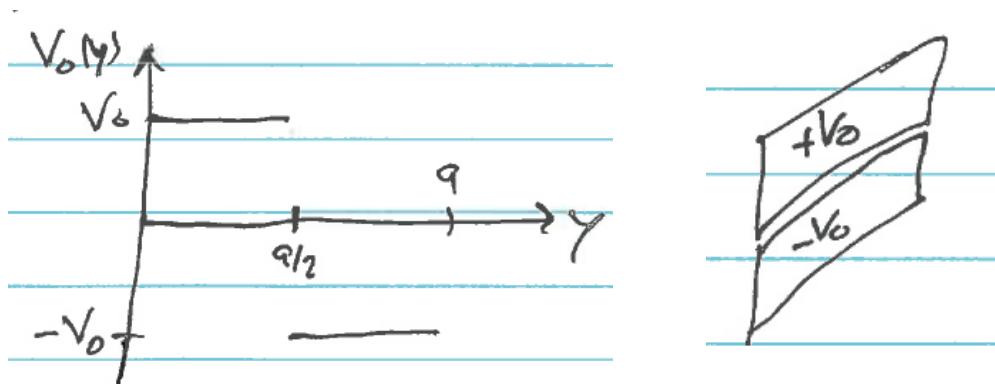
Now we have nice expression for the coefficients C_n (re-naming the dummy variable),

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

In some sense the integral projects the function $V_0(y)$ onto the basis of sine-functions $\sin\left(\frac{n\pi y}{a}\right)$.

We now have the solution for $V(x, y)$, along with all the coefficients C_n , above, for a given specified boundary condition $V_0(y)$.

The function $V_0(y)$ could take on many different forms! A specific choice of $V_0(y)$ is made in Griffiths Problem 3.15, as shown in this figure. The wall from $y = 0$ to $y = a$ is split into two plates charged to $+V_0$ and $-V_0$.



$$V_0(y) = \begin{cases} +V_0 & 0 < y < a/2 \\ -V_0 & a/2 < y < a \end{cases}$$

$$\begin{aligned} C_n &= \frac{2}{a} V_0 \left\{ \int_0^{a/2} \sin\left(\frac{n\pi y}{a}\right) dy - \int_{a/2}^a \sin\left(\frac{n\pi y}{a}\right) dy \right\} \\ &= \frac{2V_0}{a} \left\{ -\frac{\cos(n\pi y/a)}{n\pi/a} \Big|_0^{a/2} + \frac{\cos(n\pi y/a)}{(n\pi/a)} \Big|_{a/2}^a \right\} \\ &= \frac{2V_0}{n\pi} \left\{ -\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right\} \end{aligned}$$

One finds

$$C_n = \begin{cases} 8V_0/n\pi & n = 2, 6, 10, 14, \dots \\ 0 & \text{otherwise} \end{cases}$$

$C_n \neq 0$ when $n = 4j + 2$ $j = 0, 1, 2, 3, \dots$

So finally

$$V(x, y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,\dots} \frac{e^{-n\pi x/a} \sin(n\pi y/a)}{n}$$

$$V(x, y) = \frac{8V_0}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a} \sin\left[(4j+2)\frac{\pi y}{a}\right]}{4j+2}$$

Once we have $V(x, y)$, we can get

$$\vec{E} = -\vec{\nabla} V$$

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \quad \text{the surface charge density}$$

Note that in this case one should sum series first to get $V(x, y)$, then take the normal $(\partial/\partial n)$ derivative.

Take a look at the plot of $V(x, y)$ constructed using Mathematica on the class website.