

# Lecture #14 Summary

## PHYS 313 Electricity and Magnetism I

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### Another Way to Solve Laplace's Equation $\nabla^2 V = 0$

#### Separation of Variables

You can read about how to do separation of variables in Cartesian coordinates in Griffiths, section 3.3.1. It is very well written.

I am going to do separation of variables in spherical coordinates since this is the most complicated and useful approach.

We are solving  $\nabla^2 V = 0$  in a region bounded by a surface where the potential  $V$  or the charge density  $\sigma$  is specified.

The basic idea of separation of variables is to write out the full  $V$  function in terms of a product of functions, each for one variable:

$$\text{Cartesian: } V(x, y, z) = X(x) Y(y) Z(z)$$

$$\text{Cylindrical: } V(s, \phi, z) = S(s) \Phi(\phi) Z(z)$$

$$\text{Spherical: } V(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

Substitute this ansatz into  $\nabla^2 V = 0$  and “separate variables” to produce 3 ordinary differential equations from one partial differential equation.

However, there is a price to pay for assuming such an oversimplified form of the solution. That is, we generally find an *infinite* number of such solutions, and we need to include all of them to construct the correct solution that satisfies our boundary conditions. In addition, there can exist solutions to Laplace's equations that cannot be expressed in this form, and we miss these solutions altogether.

#### Spherical Coordinates

We want to solve  $\nabla^2 V = 0$  for objects with spherical boundary conditions. For example, a sphere with a potential that depends on  $\theta$  on its surface.

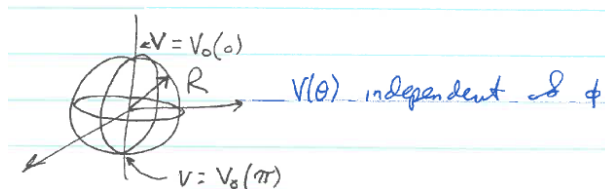


Figure 1: \*

Sphere of radius  $R$  with surface potential  $V = V_0(\theta)$ , independent of  $\phi$ .

Our objective is to solve for  $V(r, \theta, \phi)$  inside and outside this sphere, subject to the boundary condition  $V(R, \theta, \phi) = V_0(\theta)$ .

Start by writing out  $\nabla^2 V(r, \theta, \phi) = 0$  in spherical coordinates, from the inside front cover of the book:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Now, to simplify the problem, assume azimuthal symmetry. The problem is the same for all rotations about the  $z$ -axis, i.e. it is independent of  $\phi$ . This allows us to drop the last term, and after multiplying through by  $r^2$  we have:

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

## Separation of Variables in Spherical Coordinates

Now separate variables with the ansatz  $V(r, \theta) = R(r) \Theta(\theta)$ :

$$\frac{\partial}{\partial r} \left( r^2 \frac{dR}{dr} \Theta \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta R \frac{d\Theta}{d\theta} \right) = 0$$

Divide through by the product  $R(r) \Theta(\theta)$ :

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

Look at this equation carefully. It says that

$$f(r) + g(\theta) = 0$$

for all  $r$  and  $\theta$ . How can this be?  $r$  and  $\theta$  are two independent orthogonal components, and this sum can equal zero only under one very special circumstance. Both  $f(r)$  and  $g(\theta)$  must be constant:

$$f(r) = C, \quad g(\theta) = -C \quad \implies \quad f(r) + g(\theta) = C - C = 0$$

So

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = C \quad \text{and} \quad \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = -C$$

It turns out that the constant  $C$  can be written as  $C = \ell(\ell + 1)$ . [This makes the  $R(r)$  solution particularly simple.] Note that  $\ell$  is not yet determined.

## Radial Equation

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \ell(\ell + 1) R$$

or equivalently,

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \ell(\ell + 1) R = 0$$

This has two simple solutions:

$$R(r) = A r^\ell + \frac{B}{r^{\ell+1}}$$

Note:  $A r^\ell$  is regular as  $r \rightarrow 0$  but diverges as  $r \rightarrow \infty$ .  
 $B/r^{\ell+1}$  is regular as  $r \rightarrow \infty$  but diverges as  $r \rightarrow 0$ .

## Angular Equation — Legendre's Equation

Now look at the angular ( $\theta$ ) equation:

$$\frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \ell(\ell + 1) \sin \theta \Theta = 0$$

This is called Legendre's Equation. It comes up in the solution of the Schrödinger Equation for the Hydrogen Atom:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{e^2}{4\pi\epsilon_0 r} \psi = E\psi$$

Make a substitution change of variable:

$$u = \cos \theta \quad \implies \quad \sin \theta = \sqrt{1 - u^2} = \sqrt{1 - \cos^2 \theta}$$

$\theta = 0 \rightarrow \pi$  now corresponds to  $u = 1 \rightarrow -1$ . In other words the substitution  $u = \cos \theta$  maps  $\theta \in [0, \pi]$  to  $u \in [-1, 1]$ .

Also change the function from  $\Theta(\theta)$  to  $P(u)$ :

$$\Theta(\theta) = P(u)$$

So

$$\frac{d\Theta}{d\theta} = \frac{dP}{du} \frac{du}{d\theta} = -\sin \theta \frac{dP}{du} = -\sqrt{1 - u^2} \frac{dP}{du}$$

This transforms Legendre's Equation to

$$-\sqrt{1 - u^2} \frac{d}{du} \left( -\sqrt{1 - u^2} \sqrt{1 - u^2} \frac{dP}{du} \right) + \ell(\ell + 1) \sqrt{1 - u^2} P = 0$$

or

$$\boxed{\frac{d}{du} \left( (1 - u^2) \frac{dP}{du} \right) + \ell(\ell + 1) P = 0, \quad u \in [-1, 1]}$$

We demand solutions for  $P(u)$  that remain finite for  $u$  between  $-1$  and  $1$ , inclusive.

## Power Series Solution

How do we solve this equation? Try the power series solution:

$$P(u) = \sum_{n=0}^{\infty} C_n u^n = C_0 + C_1 u + C_2 u^2 + C_3 u^3 + \dots$$

Substitute this into the equation. Expanding  $(1 - u^2) \frac{d^2 P}{du^2} - 2u \frac{dP}{du} + \ell(\ell + 1)P = 0$ :

$$(1 - u^2) \sum_{n=0}^{\infty} n(n-1) C_n u^{n-2} - 2 \sum_{n=0}^{\infty} n C_n u^n + \ell(\ell + 1) \sum_{n=0}^{\infty} C_n u^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)C_{n+2} - n(n-1)C_n - 2n C_n + \ell(\ell + 1)C_n] u^n = 0$$

Note that  $n(n-1) + 2n = n(n+1)$ , so

$$\sum_{n=0}^{\infty} [(n+2)(n+1)C_{n+2} - n(n+1)C_n + \ell(\ell + 1)C_n] u^n = 0$$

This must hold for all  $u$ . The only way this convergent sum can be zero for all  $u$  is if all the coefficients are zero. This leads to a recursion relation:

$$(n+2)(n+1)C_{n+2} = [n(n+1) - \ell(\ell+1)] C_n$$

or

$$\boxed{\frac{C_{n+2}}{C_n} = \frac{n(n+1) - \ell(\ell+1)}{(n+1)(n+2)}}$$

So given  $C_0$ , this recursion relation determines  $C_2, C_4, C_6, C_8, \dots$

$$n = 0 : \quad \frac{C_2}{C_0} = \frac{0 - \ell(\ell+1)}{2} = -\frac{\ell(\ell+1)}{2}$$

Given  $C_1$ , this recursion relation determines  $C_3, C_5, C_7$ , etc.

$$n = 1 : \quad \frac{C_3}{C_1} = \frac{2 - \ell(\ell+1)}{6} = \frac{1}{3} - \frac{\ell(\ell+1)}{6}$$

So there are two independent constants,  $C_0$  and  $C_1$ , that multiply the two linearly independent solutions to the second order differential equation.

$C_0$  labels the solution with even powers of  $u$  and is therefore an even function.

$C_1$  labels the solution with odd powers of  $u$  and is therefore an odd function.

## Convergence and Termination

Again we demand that the solution be regular for all  $u$  in the interval  $[-1, 1]$ . For this to happen, the power series solution must converge for  $u = \pm 1$ . This will NOT converge at  $u = \pm 1$  if the series is infinite. Look at the terms as  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} \frac{C_{n+2}}{C_n} = \frac{n^2}{n^2} = 1$$

So the infinite series will look like

$$P(u) \approx \dots + u^{799} + u^{800} + u^{801} + \dots \Rightarrow \sum_{n=0}^{\infty} u^n = \frac{1}{1-u}, \quad \text{which diverges as } u \rightarrow 1.$$

Hence to get a finite  $P(u)$ , we need to **TERMINATE** the series. Terminate the series at  $n = n_{\max}$ , such that all subsequent terms are zero:

$$n = n_{\max} : \quad \frac{C_{n_{\max}+2}}{C_{n_{\max}}} = \frac{n_{\max}(n_{\max}+1) - \ell(\ell+1)}{(n_{\max}+1)(n_{\max}+2)} = 0$$

$$\implies n_{\max} = \ell \quad \text{does the trick!}$$

We need to make  $\ell$  a positive integer (or zero) to terminate the series and recover a finite result for  $P(u)$  and  $\Theta(\theta)$ . (The other solution  $\rightarrow -(\ell+1)$ .)

## Legendre Polynomials

The solutions to the differential equation are finite polynomials of degree  $\ell$ . They are called the **Legendre Polynomials**:

$$\boxed{\begin{array}{ll} n_{\max} = 0 = \ell : & P_0(u) = 1 \\ n_{\max} = 1 = \ell : & P_1(u) = u \\ n_{\max} = 2 : & P_2(u) = (3u^2 - 1)/2 \\ n_{\max} = 3 : & P_3(u) = (5u^3 - 3u)/2 \\ & P_4(u) = (35u^4 - 30u^2 + 3)/8 \end{array}}$$

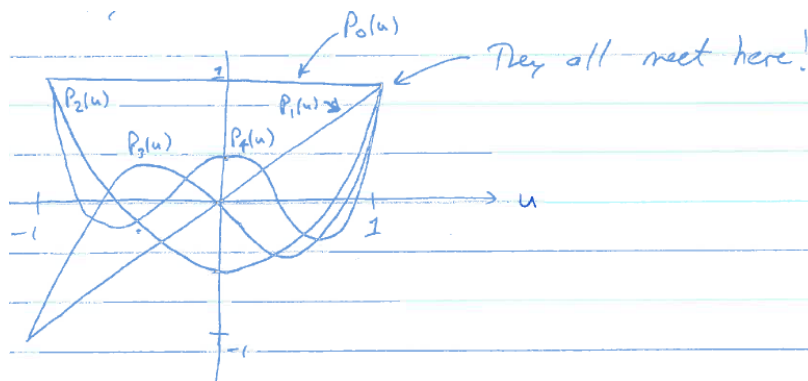


Figure 2: \*  
Legendre polynomials  $P_0$  through  $P_4$  on the interval  $[-1, 1]$ . They all meet at  $u = 1$ .

They are normalized so that  $P_\ell(1) = 1$ .

The Legendre functions constitute a complete orthonormal set on the interval  $[-1, 1]$ .

Note that the parity of  $P_\ell(u)$  is  $(-1)^\ell$ :

$$\ell \text{ even : } P_\ell(-u) = P_\ell(u)$$

$$\ell \text{ odd : } P_\ell(-u) = -P_\ell(u)$$

The Legendre Polynomials can be generated by the **Rodrigues Formula**:

$$P_\ell(u) = \frac{1}{2^\ell \ell!} \left( \frac{d}{du} \right)^\ell (u^2 - 1)^\ell$$

**Example:**  $\ell = 2 = n_{\max}$

Suppose we take  $\ell = 2 = n_{\max}$ .

$$C_0, \quad C_2 = C_0 \frac{0 - 2 \cdot 3}{1 \times 2} = -3 C_0 \quad (n = 0)$$

$$C_4 = C_2 \frac{2(3) - 2(3)}{3 \times 4} = 0, \quad C_6 = C_8 = C_{10} = \dots = 0$$

What about  $C_1$ ?

$$n = 1 : \quad C_3 = C_1 \frac{1 \times 2 - 2(3)}{2 \times 3} = -\frac{2}{3} C_1$$

$$n = 2 : \quad C_5 = C_1 \times \dots$$

A choice of  $\ell$  gives you either an even or odd solution, depending on whether  $\ell$  is even or odd.

The other solutions to the problem are called the **Legendre Function of the Second Kind**  $Q_\ell(u)$ . See E.L. Ince, *ODE*, p. 165.

When  $\ell$  is not an integer, there are infinite series that diverge at  $u = \pm 1$ , in general, so are not useful for physical problems.