

Lecture #13 Summary

PHYS 313 Electricity and Magnetism I

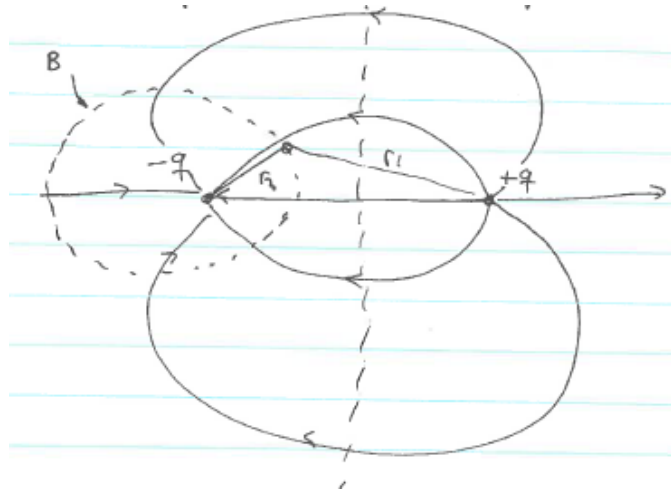
Prof. Steven Anlage

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The Method of Images

This is a technique to solve electrostatics problems involving conducting surfaces.

Consider two equal and opposite point charges a distance $2d$ apart in free space.



The electric field lines flow from $+q$ to $-q$. Perpendicular to these lines lie equipotential surfaces that satisfy

$$V = \frac{+q}{4\pi\epsilon_0 r_1} - \frac{q}{4\pi\epsilon_0 r_2} = \text{constant}$$

One such equipotential surface is the plane halfway between the two charges. All points on that plane are an equal distance from $+q$ and $-q$, and the potential is zero in that plane.

But there are many other surfaces that satisfy

$$\frac{q}{r_1} - \frac{q}{r_2} = \text{const}$$



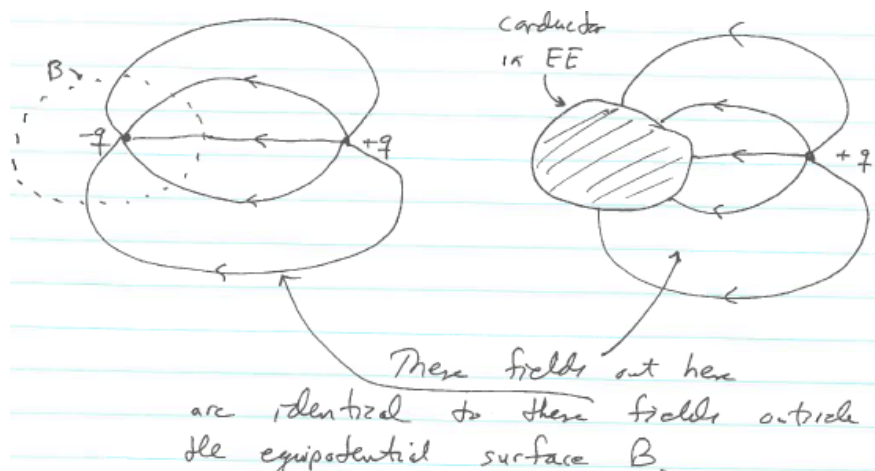
Look at surface B .

Suppose we put a thin sheet of metal so that it exactly followed surface B . It would be an equipotential surface in EE and its potential could be set to match that of equipotential surface B due to the two charges. This surface would not disturb the two charges at all!

The fields inside and outside this equipotential surface are quite independent of each other. (Think of a charge inside a spherical shell — it is “isolated” from charges outside the shell.)

Here we could fill in the interior of the conducting volume, and it would have no effect on the fields outside.

In Other Words



These fields out here are identical to these fields outside the equipotential surface B .

Or —

The two point charge problem is equivalent to the point charge near a conductor problem, at least for fields outside the conductor.

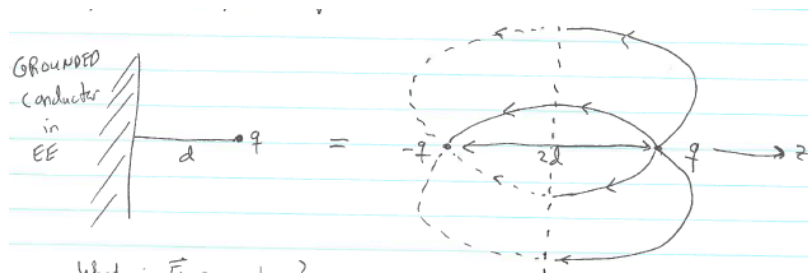
This is the basic idea of the method of images.

An image charge(s) is used to substitute/create an equipotential surface that exactly matches the surface of a conductor in EE.

Solving the point charge problem is much easier than the original problem! Note that $\vec{E}$ is $\perp$ to the conductor in EE.

Point Charge Near a Grounded ($V = 0$) Conducting Plane

The easiest problem is when we have a point charge facing a plane.



What is $\vec{E}$ everywhere? What is the induced charge on the conductor?

Notice that the image problem satisfies the same boundary conditions for the potential:

- (a) $V = 0$ on the plane, since $\frac{+q}{r_1} - \frac{q}{r_2} = 0$ on the plane.
- (b) $V(\infty) \rightarrow 0$

The potential at a point (x, y, z) in space can be simply written as

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left\{ \underbrace{\frac{+q}{\sqrt{x^2 + y^2 + (z-d)^2}}}_{r_1} - \underbrace{\frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}}_{r_2} \right\}$$

Note that this only works for $z > 0$. For $z < 0$, $V(x, y, z < 0) = 0$.

This solution is correct and unique. It satisfies Laplace's Equation in the region $z > 0$ and Poisson's Equation $\nabla^2 V = -\rho/\epsilon_0$, and it satisfies the imposed boundary conditions $V = 0$ on the surface and $V \rightarrow 0$ as $r \rightarrow \infty$. Hence by the first uniqueness theorem, it is *the* answer to Laplace's Eq.

We can also use the method of images for magneto-static problems.

Induced Surface Charge

We can also find the charge induced on the surface of the conductor due to the presence of the charge.

The field just outside a conductor is like that produced by a "one-sided" charged plane.



Use Gauss's Law on the pill-box bisected by the conducting surface:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$E(h) A = \frac{\sigma A}{\epsilon_0}$$

$$\vec{E}(h) = \frac{\sigma}{\epsilon_0} \hat{n}$$

where $\hat{n}$ is the outward normal to the conductor surface.

Hence the normal component of the electric field at the surface is directly proportional to the surface charge density:

$$\sigma = \epsilon_0 \vec{E} \cdot \hat{n} = -\epsilon_0 \frac{\partial V}{\partial n}$$

where n is the normal coordinate.

In our case the normal direction is the $+z$ direction:

$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0}$$

From our solution to $V(x, y, z)$, we have:

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{-q(z-d)}{(x^2 + y^2 + (z-d)^2)^{3/2}} + \frac{q(z+d)}{(x^2 + y^2 + (z+d)^2)^{3/2}} \right\}$$

$$\left. \frac{\partial V}{\partial z} \right|_{z=0} = \frac{1}{4\pi\epsilon_0} \frac{2qd}{(x^2 + y^2 + d^2)^{3/2}}$$

So

$$\sigma = -\frac{\epsilon_0}{4\pi\epsilon_0} \frac{2qd}{(x^2 + y^2 + d^2)^{3/2}}$$

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}$$

So the induced charge is largest just beneath the point charge ($x = y = 0$) and falls off in any direction from there.

If this induced charge is integrated over the whole (x, y) plane, the total charge is $-q$:

$$Q = \int_0^{2\pi} \int_0^\infty \frac{-qd}{2\pi(r^2 + d^2)^{3/2}} r dr d\phi = -q$$

Carrying out the integration:

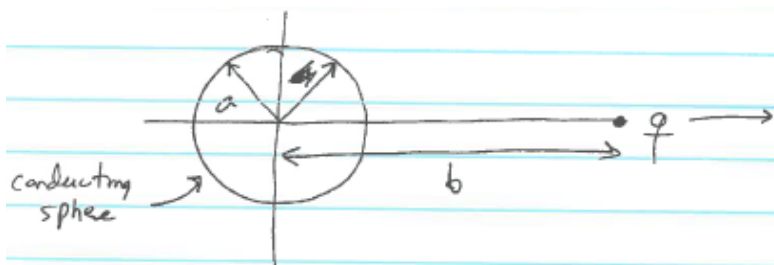
$$Q = \frac{-qd}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty \frac{r}{(r^2 + d^2)^{3/2}} dr$$

Let $u = r^2 + d^2$, $du = 2r dr$:

$$\begin{aligned} &= -qd \int \frac{1}{2} \frac{du}{u^{3/2}} \\ &= -\frac{qd}{2} (-2) u^{-1/2} \Big|_{d^2}^\infty \\ &= qd \left[\frac{1}{\sqrt{r^2 + d^2}} \right]_{d^2}^\infty = qd \left[0 - \frac{1}{d} \right] \\ &Q = -q \end{aligned}$$

Hence all the field lines emitted by $+q$ are intercepted by negative charges on the surface of the conductor before they can enter the metal!

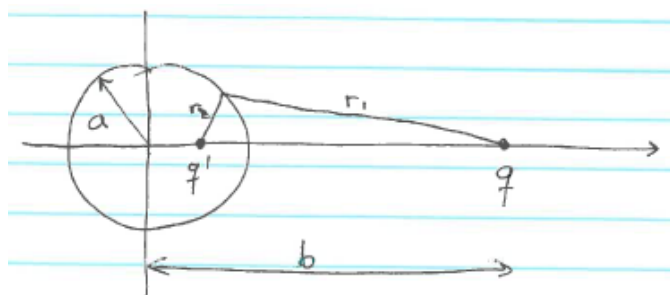
Method of Images: Point Charge Near a Conducting Sphere



Can we find an “image charge” that will make the sphere an equipotential surface for those two charges? The answer is yes!

Take the sphere to be of radius a , with a center b away from the charge.

It turns out that the image charge is not at the center of the sphere and is not of equal and opposite charge. Put it on the axis and call it q' .



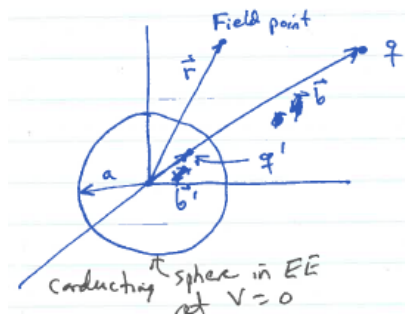
Point Charge in the Presence of a Grounded Sphere

Sphere of radius a at the origin. Point charge q a distance b from center of the sphere. The sphere of radius a is grounded ($V = 0$). Find $\vec{E}$ everywhere.

We expect the image charge q' at $\vec{b}'$ will be on the same line from the origin as q .

The total potential at a field point $\vec{r}$ outside the sphere is:

$$V(\vec{r}) = \frac{q/4\pi\epsilon_0}{|\vec{r} - \vec{b}|} + \frac{q'/4\pi\epsilon_0}{|\vec{r} - \vec{b}'|}$$



Now choose q' and $\vec{b}'$ so that $V(|\vec{r}| = a) = 0$.

Let $\hat{r}$ be the unit vector in the direction of $\vec{r}$. Let $\hat{r}'$ be a unit vector in the direction of $\vec{b}'$. Then

$$V(\vec{r}) = \frac{q/4\pi\epsilon_0}{|r\hat{r} - b'\hat{r}'|} + \frac{q'/4\pi\epsilon_0}{|r\hat{r} - b'\hat{r}'|}$$

Factor r out of the first term and b' out of the second term:

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r} \frac{1}{|\hat{r} - (b'/r)\hat{r}'|} + \frac{q'}{4\pi\epsilon_0 b'} \frac{1}{|\hat{r}' - (r/b')\hat{r}|}$$

Now make this equal to zero at $r = a$:

$$V(|\vec{r}| = a) = 0 = \frac{q}{a} \frac{1}{|\hat{r} - (b/a)\hat{r}'|} + \frac{q'}{b'} \frac{1}{|\hat{r}' - (a/b')\hat{r}|}$$

We have 2 degrees of freedom (q' and b') to make these two terms add up to zero. Clearly we can do this if:

$$\frac{q}{a} = -\frac{q'}{b'} \quad \text{and} \quad \frac{b}{a} = \frac{a}{b'}$$

Using the identity $|\vec{x} - \vec{y}| = \sqrt{(\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})} = \sqrt{x^2 - 2\vec{x} \cdot \vec{y} + y^2}$:

$$\left| \hat{r} - \frac{b}{a} \hat{r}' \right| = \sqrt{1 - 2 \frac{b}{a} \hat{r} \cdot \hat{r}' + \left(\frac{b}{a} \right)^2}$$

$$\left| \hat{r}' - \frac{b}{a} \hat{r} \right| = \sqrt{1 - 2 \frac{b}{a} \hat{r}' \cdot \hat{r} + \left(\frac{b}{a} \right)^2}$$

These are equal for any $\hat{r}$ and $\hat{r}'$!

So we can have our cake and eat it too:

$$q' = -\frac{b'}{a} q, \quad b' = \frac{a^2}{b}$$

$$q' = -\frac{a}{b} q \quad \text{and} \quad b' = \frac{a^2}{b}$$

Note $b' = a^2/b = a(a/b) < a$, so the image charge is inside the sphere.

The problem works in reverse too. Consider charge q' inside the sphere; q is then the image charge for calculating fields inside the sphere.

The surface of the sphere should satisfy

$$\frac{q}{r_1} + \frac{q'}{r_2} = 0$$

so

$$\frac{q'}{r_2} = -\frac{q}{r_1} \quad \text{or} \quad \frac{r_2}{r_1} = -\frac{q'}{q}$$

It turns out that if the charge $q' = -(a/b)q$ is placed a distance a^2/b from the center, you satisfy the condition on the sphere.

The potential of the sphere can be adjusted by adding an appropriately sized charge q'' to the center of the sphere, thus adding potential

$$\frac{q''}{4\pi\epsilon_0 a}$$

to the potential on the surface.

Hence the potential outside the sphere is given by

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r_1} + \frac{q'}{4\pi\epsilon_0 r_2} + \frac{q''}{4\pi\epsilon_0 r}$$