

Lecture #11 Summary

PHYS 313 Electricity and Magnetism I

Prof. Steven Anlage

Fall 2026

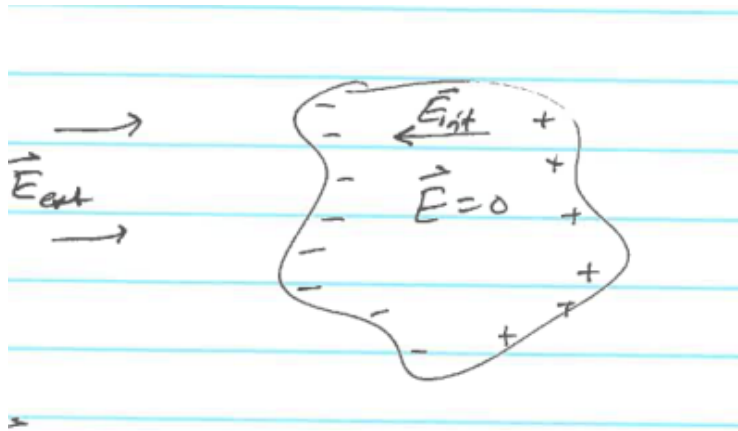
Conductors and Electrostatic Equilibrium

Conductors – A material with electrons free to move about the interior and on the surface.

Insulators – Materials in which the electrons are all bound and are not free to move.

A conductor is a perfect di-electric: dia-means “to oppose.”

A metal will re-distribute charge to eliminate any electric field in its interior.



①. $\vec{E} = 0$ inside a conductor in **Electrostatic Equilibrium (EE)**.

Charges are static (no charges move) in EE.

If there was free charge inside and $\vec{E} \neq 0$, the charge would be accelerated and the EE condition violated.

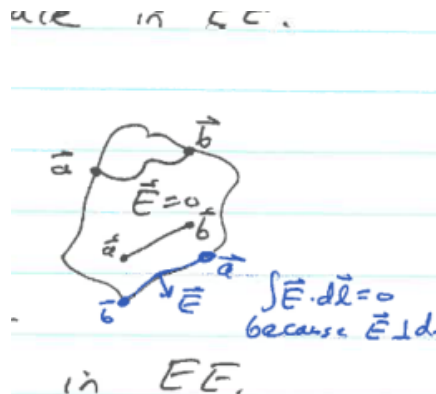
The charges re-distribute to create an internal field $\vec{E}_{\text{int}}$ that exactly cancels $\vec{E}_{\text{ext}}$:

$$\vec{E} = \vec{E}_{\text{ext}} + \vec{E}_{\text{int}} = 0 \quad \text{inside a metal in EE.}$$

②. Free charge density $\rho = 0$ inside a conductor in EE.

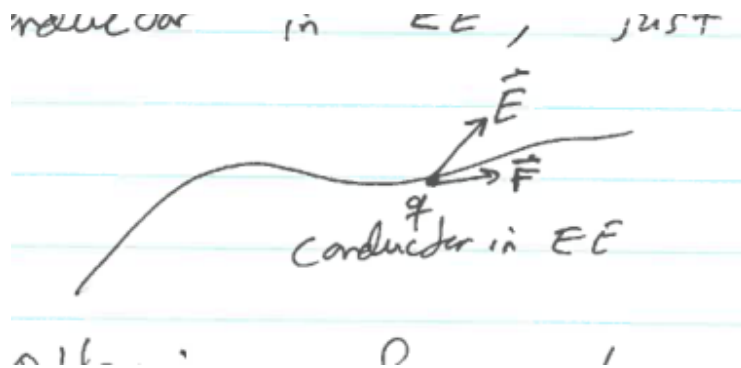
$$\vec{E} = 0 \quad \text{and} \quad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \text{hence} \quad \rho = 0 \quad \text{inside a conductor in EE.}$$

③. All net charge resides on the surface of a conductor in EE.



But $\vec{E} = 0$ inside, so $V(\vec{b}) = V(\vec{a})$ for any pair of points inside or on the surface of a conductor in EE.

⑤. $\vec{E}$ is perpendicular to the surface of a conductor in EE, just outside the conductor.

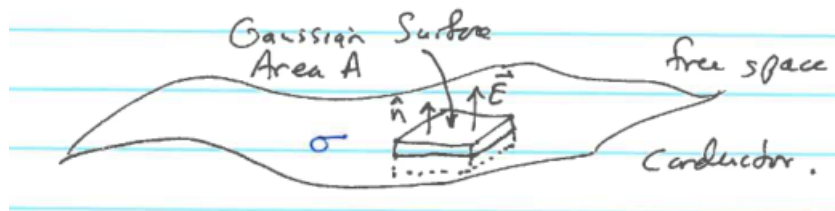


Otherwise, surface charges would be accelerated along the surface, violating the EE condition. The electric field just outside the surface is given by

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$$

where σ is the local charge per unit area on the surface.

This can be obtained from Gauss' Law applied to a small area of the surface.



$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$(\vec{E}_{\text{outside}} \cdot \hat{n})(A\hat{n}) + (\vec{E}_{\text{inside}} \cdot (-\hat{n}))(-A\hat{n}) = \frac{\sigma A}{\epsilon_0}$$

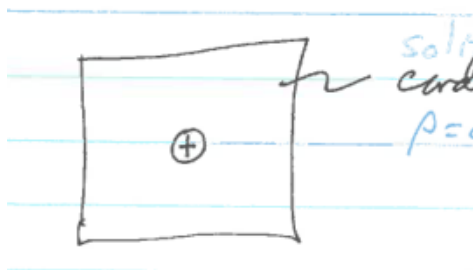
$$+4 \times 0 \quad \text{since } \vec{E} \perp \hat{n} \text{ on the four sidewalls of the Gaussian surface}$$

$$E_{\text{outside}} = \frac{\sigma}{\epsilon_0} \quad \text{so} \quad \vec{E}_{\text{outside}} = \frac{\sigma}{\epsilon_0} \hat{n}$$

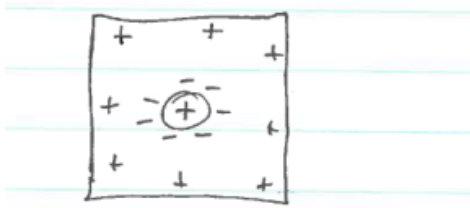
Screening of a Charge in a Metal

The dielectric property of a conductor in EE is not perfect. There is a finite penetration length of electric field inside the surface of a conductor.

Consider the following exercise: Introduce a positive charge $+q$ into an initially uncharged metal.



The charges in the metal will re-arrange to screen this charge and make $\vec{E} = 0$ inside the conductor.



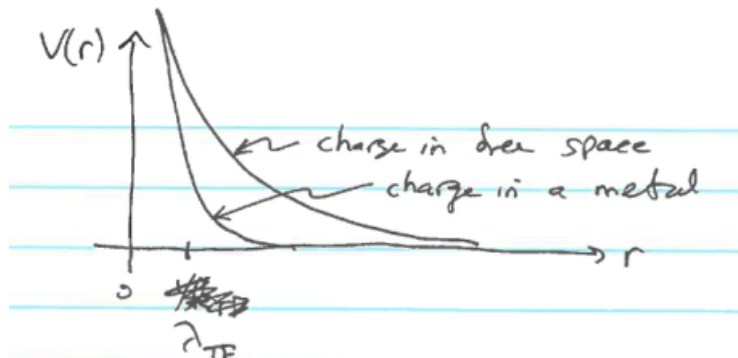
One can calculate the potential of this excess charge (at the origin of a spherical coordinate system) in the conductor using electrostatics and some quantum mechanics. The result is

$$V(r) = \frac{q}{4\pi\epsilon_0} \frac{e^{-r/\lambda_{TF}}}{r}$$

as opposed to

$$V(r) = \frac{q}{4\pi\epsilon_0 r}$$

for a charge q in free space.



where the new length scale is

$$\lambda_{TF} = \sqrt{\frac{2\epsilon_0 E_F}{3\rho e}}$$

E_F = Fermi Energy

$\rho = ne$ = charge per unit volume of “free electrons”

n = charge density of “free” electrons

For Aluminum

$$E_F = 11.63 \text{ eV}$$

$$n = 18 \times 10^{22} / \text{cm}^3$$

$$\rho = ne = 2.88 \times 10^4 \text{ C/cm}^3$$

$$\lambda_{TF} = 0.5 \text{ \AA} \simeq a_{\text{Bohr}}$$

So the Thomas-Fermi screening length is on the order of an atomic radius!



Other techniques to calculate Electric Potential due to a Static Charge Distribution

First, let's return to the neglected Poisson Equation

$$\vec{E} = -\nabla V$$

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\nabla \cdot (-\nabla V) = \rho/\epsilon_0 \quad \text{or} \quad \nabla^2 V = -\rho/\epsilon_0$$

In the special case where no charge is located in the region of interest, we have Laplace's equation:

$$\nabla^2 V = 0$$

Now comes a couple of important questions:

①. How do we solve this equation for V ?

and a related question:

②. What conditions are required to uniquely determine V ?

Clearly, we must supply Boundary Conditions (BCs) to solve Laplace's Eqn:

$$\nabla^2 V = 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad V(x, y, z)$$

This is a partial differential equation.

V must be specified on some boundary, and/or derivatives of V must be specified.

General characteristics of solutions to Laplace's equation

1. The value of the potential V at any point $\vec{r}$ is the *average value* of V over a spherical surface of radius R centered at $\vec{r}$.

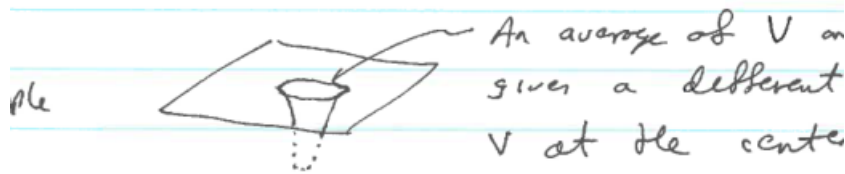
$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V da$$

This is true for any R (until you run into a conductor or some charge)!

This statement is proven in Griffiths by considering the potential due to a point charge, and then using superposition to extend the proof to any distribution of charge.

This idea is the basis of the relaxation method used to solve $\nabla^2 V = 0$ on a computer. See "Lecture 11 Supplement on solutions to Laplace's equation".

2. As a consequence of 1, V can have no local maxima or minima. All the extreme values of V occur on the boundaries.



3. Uniqueness Theorem #1

The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface S .

This is an enormously useful theorem. If you find any solution that satisfies $\nabla^2 V = 0$ and V specified on S , then you have found the unique solution!

4. Uniqueness Theorem #1.5

The potential in a volume $\mathcal{V}$ is uniquely determined if a) the charge density throughout the volume, and b) the value of V on the boundary, are specified.

$$\nabla^2 V = -\rho/\epsilon_0$$

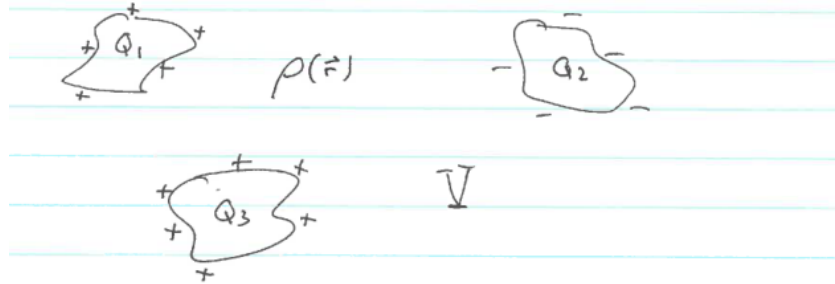
5. Uniqueness Theorem #2

Many electrostatic problems come in this form

a) Specify the charge on a series of conductors

b) Specify charge density $\rho(\vec{r})$ in the region in between the conductors

The potential is not known on these conductors, so Uniqueness Theorem #1 may not apply.



Now solve for V :

$$\nabla^2 V = -\rho/\epsilon_0$$

with the charged conductors. Is there a unique solution for $\vec{E}$?

Yes, In a volume $\mathcal{V}$ containing conductors and a specified charge density ρ , the electric field is uniquely determined if the total charge on each conductor is given.

The volume $\mathcal{V}$ can be bounded by another conductor, or be unbounded.