

Lecture #10 Summary

PHYS 313 Electricity and Magnetism I

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Work and Energy in Electrostatics

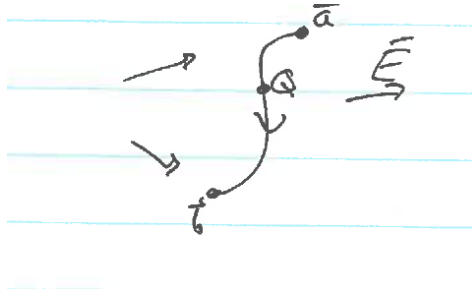
Notation: following Griffiths, the separation (relative) vector between two points is denoted

$$\vec{z} = \vec{r} - \vec{r}'$$

and subscripted versions such as z_{ij} denote the distance between charges i and j .

How much energy is stored in a static configuration of charge?

First calculate the work done on a test charge Q moved through a region of space with an electric field $\vec{E}$ is present.



The test charge experiences a force $\vec{F} = Q\vec{E}$.

To complete this displacement infinitesimally slowly, an equal and opposite force $\vec{F}_{\text{apply}} = -\vec{F}$ must be applied to the test charge.

The work that you do to complete this displacement is:

$$\begin{aligned} W &= \int_{\vec{a}}^{\vec{b}} \vec{F}_{\text{apply}} \cdot d\vec{\ell} \\ &= -Q \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell} \end{aligned}$$

But we found in Lecture 7 that the potential difference is defined as:

$$V(\vec{b}) - V(\vec{a}) = - \int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{\ell}$$

Hence

$$W = Q[V(\vec{b}) - V(\vec{a})]$$

So the change in electric potential is related to the work done against the electric field divided by the test charge magnitude

$$V(\vec{b}) - V(\vec{a}) = \frac{W}{Q}$$

In other words, the work done to move a charge from very far away and stick it at point $\vec{r}$ is

$$W = Q[V(\vec{r}) - V(\infty)]$$

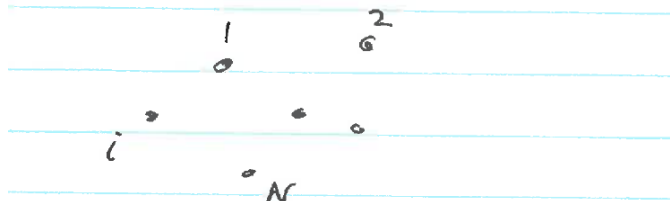
$$W = Q V(\vec{r}) \quad \text{with the convention that the reference point at infinity has } V(\infty) = 0.$$

So the electric potential is the electrostatic potential energy per unit charge.

$$“V = \frac{W}{Q}” \quad \text{like} \quad “\vec{E} = \frac{\vec{F}}{Q}”$$

Given this connection between electric potential and electrostatic energy, we can now calculate the energy stored in static charge configurations.

Suppose we have N static charges (imagine the 92 protons in a Uranium nucleus, for example).



How much energy is stored in this configuration?

⇒ Solution.

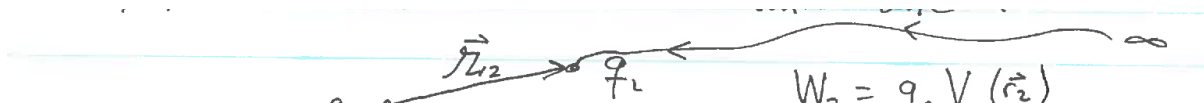
Imagine building it up one charge at a time and add up all the work it takes to complete the job.

Start with empty space and bring in the charges one by one.

(1) The first charge requires no work to bring in from ∞ .

$$W_1 = 0 \quad q_1$$

(2) The second charge must fight against (assuming equal signs for the charges) the first. The total work done is

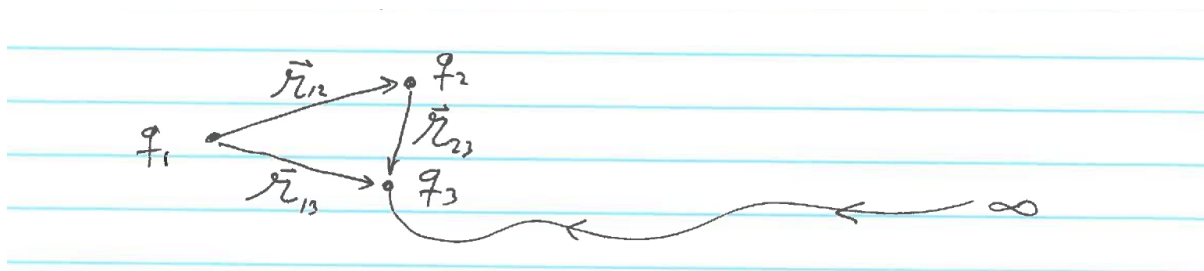


$$W_2 = q_2 V_1(\vec{r}_2)$$

or using the fact that $V(r) = \frac{q}{4\pi\epsilon_0 r}$ for a point charge:

$$W_2 = q_2 \frac{q_1}{4\pi\epsilon_0 |\vec{r}_{12}|}$$

(3) The third charge must now fight the first two:



$$\begin{aligned} W_3 &= q_3 V_1(\vec{r}_3) + q_3 V_2(\vec{r}_3) \quad \text{superposition of } V \\ &= q_3 \frac{q_1}{4\pi\epsilon_0 |\vec{r}_{13}|} + q_3 \frac{q_2}{4\pi\epsilon_0 |\vec{r}_{23}|} \end{aligned}$$

(4) The fourth charge now fights three others to get into position:

$$W_4 = q_4 V_1(\vec{r}_4) + q_4 V_2(\vec{r}_4) + q_4 V_3(\vec{r}_4)$$

$$= q_4 \frac{q_1}{4\pi\epsilon_0 |\vec{r}_{14}|} + q_4 \frac{q_2}{4\pi\epsilon_0 |\vec{r}_{24}|} + q_4 \frac{q_3}{4\pi\epsilon_0 |\vec{r}_{34}|}$$

The total work up to this point is

$$W = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}} \right)$$

A trend is emerging here. In general, for N charges, one has

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \sum_{j>i}^N \frac{q_i q_j}{r_{ij}}$$

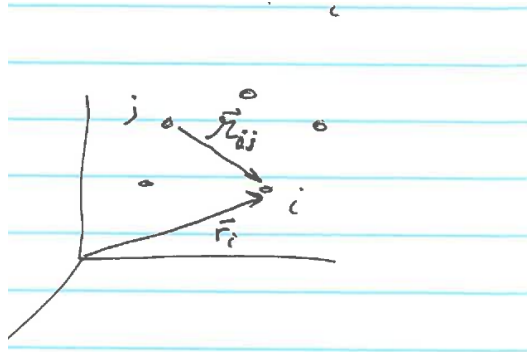
The $j > i$ sum avoids double counting. You can intentionally double count, and then correct by multiplying by $1/2$:

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_i q_j}{r_{ij}}$$

$j \neq i$ avoids the embarrassing fact that the self energy of an electron (or point charge) is infinite! Re-write the sum as follows:

$$W = \frac{1}{2} \sum_{i=1}^N q_i \left(\sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_j}{4\pi\epsilon_0 r_{ij}} \right)$$

This last term (in parentheses) is the potential produced by the $N - 1$ other charges at $\vec{r}_i$.



$$\sum_{\substack{j=1 \\ j \neq i}}^N \frac{q_j}{4\pi\epsilon_0 r_{ij}} = V(\vec{r}_i)$$

So

$$\boxed{W = \frac{1}{2} \sum_{i=1}^N q_i V(\vec{r}_i)} \quad (1)$$

This is how much work it takes to assemble the N charges.

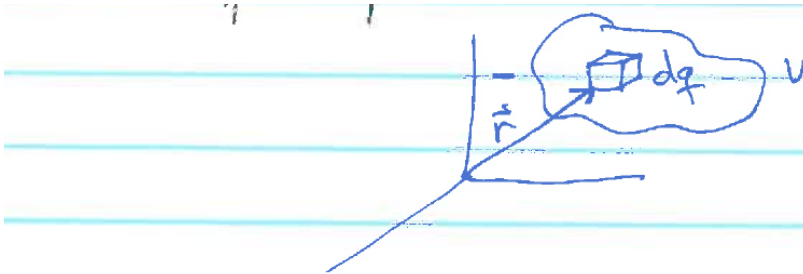
Now suppose there is a continuous distribution of charges. Then

$$W = \frac{1}{2} \int_{Vol} dq V(\vec{r})$$

and, as before,

$$dq = \rho d\tau' \quad \text{or} \quad dq = \sigma da' \quad \text{or} \quad dq = \lambda d\ell'$$

Subtlety: In the equation above $V(\vec{r}_i)$ is the potential due to all the *other* point charges. Whereas now $V(\vec{r})$ refers to the total potential produced by *all* the charge. This now includes the self energy of *all* the charges, whereas Eq. (1) above does not. This difference has consequences!



For a 3D charge distribution, this becomes

$$W = \frac{1}{2} \int_{Vol} V(\vec{r}') \rho(\vec{r}') d\tau'$$

This form is not always convenient to use, mostly because it involves yet another integral over all the source charge. This can be avoided by using the Divergence Theorem.

First use the fact that $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, so

$$W = \frac{\epsilon_0}{2} \int_{Vol} V(\vec{r}') \vec{\nabla} \cdot \vec{E} d\tau$$

We can integrate by parts. Consider this product rule

$$\vec{\nabla} \cdot (V\vec{E}) = V \vec{\nabla} \cdot \vec{E} + \vec{E} \cdot \vec{\nabla} V$$

Integrate both sides over a volume $\mathcal{V}$

$$\int_{\mathcal{V}} \vec{\nabla} \cdot (V\vec{E}) d\tau = \int_{\mathcal{V}} V \vec{\nabla} \cdot \vec{E} d\tau + \int_{\mathcal{V}} \vec{E} \cdot \vec{\nabla} V d\tau$$

Apply the divergence theorem to the LHS:

$$\int_{\mathcal{V}} \vec{\nabla} \cdot (V\vec{E}) d\tau = \oint_S (V\vec{E}) \cdot d\vec{a} \quad \text{where } S \text{ encloses } V$$

or, solving for the volume integral of interest:

$$\int_{Vol} V \vec{\nabla} \cdot \vec{E} d\tau = \oint_S V\vec{E} \cdot d\vec{a} - \int_{Vol} \vec{E} \cdot \vec{\nabla} V d\tau$$

So finally:

$$W = \frac{\epsilon_0}{2} \left(- \int_{Vol} \vec{E} \cdot \vec{\nabla} V d\tau + \oint_S V\vec{E} \cdot d\vec{a} \right)$$

But $\vec{E} = -\vec{\nabla} V$, so

$$W = \frac{\epsilon_0}{2} \left(\int_{Vol} E^2 d\tau + \oint_S V\vec{E} \cdot d\vec{a} \right)$$

The volume Vol should include all the charge in the problem (since we started with $W = \int V\rho d\tau'$), but any larger surface will also work since $\rho = 0$ at locations beyond the (assumed finite) charge distribution.

Suppose we make the volume $\mathcal{V}$, and the area S , get larger. The E^2 integral must only get bigger since the integrand is positive. The surface integral must get smaller to balance out. It should go as

$$\oint_S V\vec{E} \cdot d\vec{a} \sim \frac{1}{r} \frac{1}{r^2} r^2 \sim \frac{1}{r} \quad \text{for point charges.}$$

Everything looks like a point charge if you get far enough away from it! This excludes those artificial problems of infinite surface and infinite line charges.

Suppose the volume expands to fill all space. Then we have

$$W = \frac{\epsilon_0}{2} \int_{\text{All space}} E^2 d\tau$$

$\Rightarrow$ From this we conclude that there is an energy density associated with the electric field

$$u_E = \frac{1}{2} \epsilon_0 E^2 \quad (\text{J/m}^3)$$

You are doing a total energy calculation for HW Griffiths 2.35

Total Electrostatic Energy

Consider the difference between

$$W = \frac{1}{2} \sum_{i=1}^N q_i V(\vec{r}_i) \quad (2)$$

(potential due to the other $N - 1$ charges produced at the site of q_i) and

$$W = \frac{1}{2} \int \rho V d\tau \quad (3)$$

$$= \frac{\epsilon_0}{2} \int_{\text{All space}} E^2 d\tau \quad (4)$$

(1) Take a single point charge $\rho = q \delta^3(\vec{r})$

Eq. (2) gives $W = 0$ since no work is done to bring a point charge from infinity to a region free of charge.

Eq. (3) gives

$$\frac{1}{2} \int q \delta^3(\vec{r}) \frac{q}{4\pi\epsilon_0 r} d\tau = \infty \quad (\text{The delta function picks out the function at the point where it's arg} = 0)$$

(Note that we can increase the volume of integration to all space since $\rho = 0$ there anyway.)

Eq. (4) gives

$$\begin{aligned} & \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{q^2}{16\pi^2 \epsilon_0^2 r^4} r^2 dr \sin \theta d\theta d\phi \\ &= \frac{q^2}{32\pi^2 \epsilon_0} 4\pi \int_0^\infty \frac{dr}{r^2} \\ &= -\frac{1}{r} \Big|_0^\infty \\ &= \infty \end{aligned}$$

(2) With two point charges, it's no better

Eq. (2) gives

$$\begin{aligned} W &= \frac{1}{2} \left(\frac{q_1 q_2}{4\pi\epsilon_0 z_{12}} + \frac{q_2 q_1}{4\pi\epsilon_0 z_{21}} \right) \quad \text{sum } i = 1 \text{ to } 2 \\ W &= \frac{q_1 q_2}{4\pi\epsilon_0 z_{12}} \end{aligned}$$

Eq. (3) gives ∞

Eq. (4) gives

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \iiint \left(\frac{q_1^2}{16\pi^2 \epsilon_0^2 |\vec{r} - \vec{r}_1|^2} + \frac{q_2^2}{16\pi^2 \epsilon_0^2 |\vec{r} - \vec{r}_2|^2} \right) d\tau \\ &= \infty \quad \text{as the integration point sweeps through } \vec{r}_1 \text{ and } \vec{r}_2. \end{aligned}$$

So the two ways of energy calculation Eq. (2) and Eqs. (3), (4) account for different parts of the energy.

Eqs. (3) and (4) include the self energy of the charges, while Eq. (2) does not. Eq. (2) counts only the energy to assemble the charges from the reference point at ∞ to their final configuration.