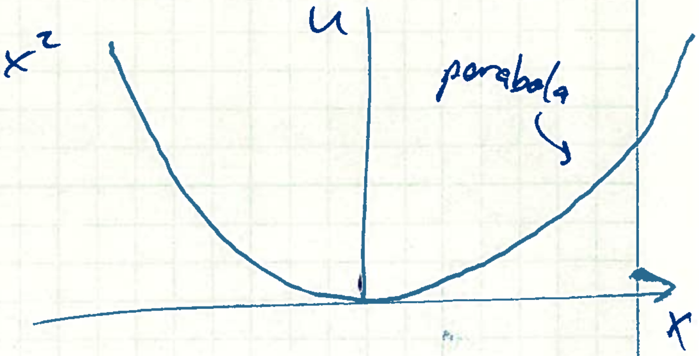
Exam 1 Review

Simple Harmonic Oscillator

(no damping)
(no driving force)

$F = -kx \Rightarrow U = \frac{1}{2}kx^2$



Eq. of motion:

$$\ddot{x} + \frac{k}{m}x = 0$$

Define $\omega_0^2 = k/m$

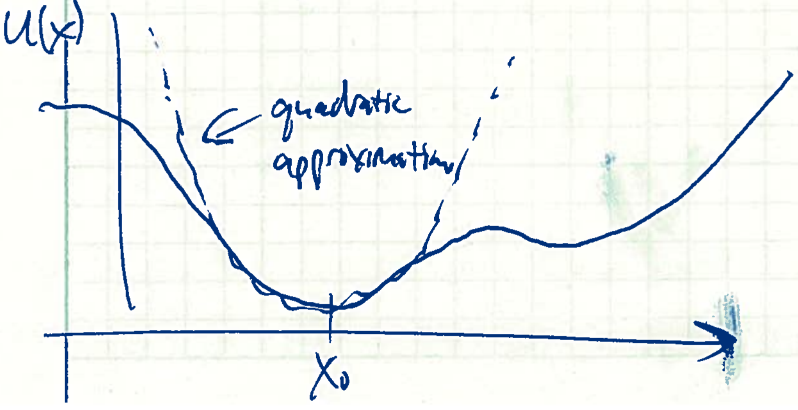
Solution: $x = Ae^{i(\omega_0 t+\delta)}$, A, δ determined by initial conditions

$\uparrow$ natural frequency.

Small oscillations: Near a ~~s~~ stable equilibrium, almost any potential function is approximately quadratic.

$$\underset{\substack{\uparrow \\ \text{equilibrium}}}{U(x_0}+\underset{\substack{\uparrow \\ \text{displacement}}}{a)} \approx U(x_0) + \underset{\substack{\uparrow \\ U'(x_0) = 0}}{U'(x_0)a} + \frac{1}{2}U''(x_0)a^2 + \dots$$

$U'(x_0) = 0$ since x_0 is an equilibrium point.



Frequency of small oscillations: $\omega_0 \approx \sqrt{\frac{U''(x_0)}{m}}$

near an equilibrium.

Plane Pendulum: $U = mgl(1-\cos\theta)$

$$\Rightarrow \omega_0 = \sqrt{g/l}$$

Complex Numbers

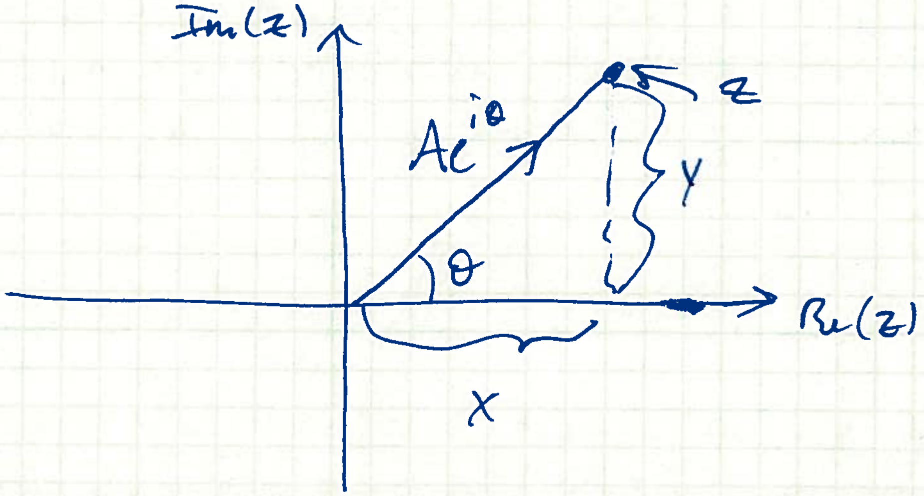
$$z = x+iy = Ae^{i\theta} ,$$

$A = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

& $x = A\cos\theta$, $y = A\sin\theta$

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$ Euler Formula

- $x = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $y = \frac{e^{i\theta}-e^{-i\theta}}{2}$



$z^* \equiv$ complex conjugate $= x - iy$ (if $z = x + iy$)

$= Ae^{-i\theta}$ (if $z = Ae^{i\theta}$)

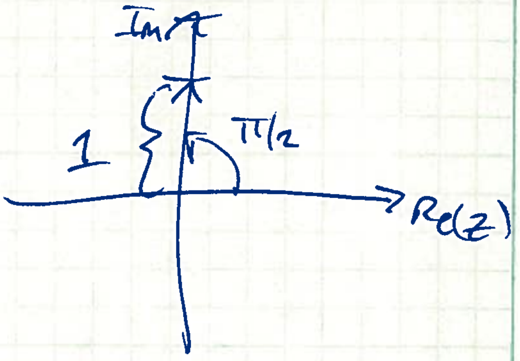
Division:

$$\frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

Also:

$$zz^* = (x+iy)(x-iy) = x^2+y^2 = A^2.$$

Also: $e^{i\pi/2} = i$



Also: $e^{i\pi} = -1$

Also: $e^{i\pi} + 1 = 0$ ← nice algebraic equation.

Forced Oscillator

Eq of Motion

$$\ddot{x} + \omega_0^2 x = \frac{F(t)}{m}$$

If $F(t) = F_0 e^{i\omega t}$

$$\ddot{x} + \underset{\substack{\uparrow \\ \text{“natural} \\ \text{frequency”}}}{\omega_0^2} x = \frac{F_0}{m} e^{i\omega t} \quad \uparrow \text{“driving frequency”}$$

Also: Add damping: $F_{drag} = -bv = -\underset{\substack{\uparrow \\ \text{some constant}}}{b}\dot{x}$

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}}$$

where $\gamma \equiv \frac{b}{m}$

$\omega_0^2 = k/m$

~~Solutions~~

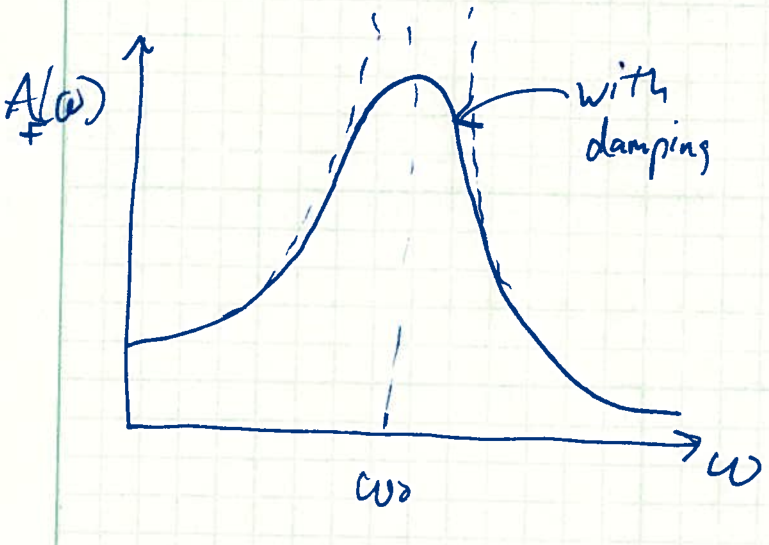
Steady-State (Long-term) Solution:

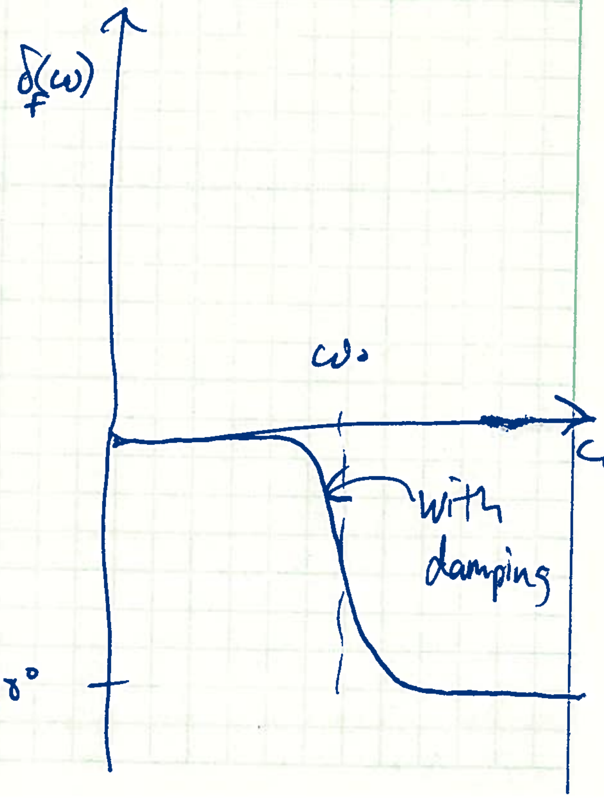
$$x(t) = A_f(\omega_f)\, e^{i(\omega_f t + \delta_f(\omega_f))}$$, $\boxed{\text{no free parameters}}$

where $A_f(\omega) = \dfrac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}}$

and $\delta_f(\omega) = -\tan^{-1}\left[\dfrac{\omega\gamma}{(\omega_0^2-\omega^2)}\right]$

ω_f = driving frequency





(Short-term)

Including the transient Solution (Short-term Behavior):

$$x(t) = A(\omega_F) e^{i(\omega_F t + \delta_F(\omega_F))} + B e^{-\gamma t/2} e^{i(\omega_d t + \delta_d)}$$

ω_d = damped frequency $= \sqrt{\omega_0^2 - \gamma^2/4}$

B & δ_d = Free parameters, determined by initial conditions.

Special Case: Damped oscillator, no forcing function

Then $F_0 = 0$, $A(\omega_F) = 0$,

and $x(t) = B \underbrace{e^{-\gamma t/2}}_{\substack{\uparrow \\ \text{damping factor}}} e^{i(\omega_d t + \delta_d)}$

Energy

- Mechanical Oscillators: $KE = \frac{1}{2} m \dot{x}^2$

  $U = \frac{1}{2} k x^2$

- Electrical Oscillators: $U_E = \int_{\text{all space}} \frac{1}{2} \varepsilon_0 |\vec{E}| \, dV$

  for capacitors $\rightarrow$ $= \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{Q^2}{2C}$

  $U_B = \int_{\text{all space}} \frac{1}{2\mu_0} |\vec{B}|^2 dV$

  for inductors $\longrightarrow$ $= \frac{1}{2} L I^2$

Energy loss

$Q \equiv \frac{\omega_0}{\gamma}$ = unitless & very large for lightly damped oscillators.

For lightly damped oscillators

$$Q = \frac{1}{\text{Fraction of energy lost in time } t = \frac{1}{\omega_0}} = \frac{2\pi}{\text{fraction of energy loss in one period}}$$

Energy loss: $E(t) = E_0 e^{-\gamma t} = KE(t) + U(t)$ for mechanical oscillators

$= U_E(t) + U_B(t)$ for electrical oscillators

AC circuits

Voltage rules:

$|V_C| = |\frac{1}{C} Q|$ Capaciter

$|V_L| = |L \frac{dI}{dt}|$ Inducter

$|V_R| = |IR|$ Resister

Simple LC circuit: $\omega_0 = \frac{1}{\sqrt{LC}}$ (simple harmonic oscillator)

Impedances:

$Z_R = R$

$Z_L = i\omega L$

$Z_C = \frac{-i}{\omega C}$

Series Combination: $Z_{series} = Z_1 + Z_2$

Parallel Combination: $Z_{parallel} = \left[(Z_1)^{-1} + (Z_2)^{-1}\right]^{-1}$

For any element or combination of elements,

$$\vec{V} = \vec{I} Z \leftarrow \text{impedance, possibly complex.}$$

Voltage phasor $\uparrow$ ($\vec{V}$)

$\uparrow$ current phase ($\vec{I}$)

~~If Z~~ In general Z is complex, which means that there is a phase offset between $\vec{V}$ & $\vec{I}$.

The ratio of $\frac{V_0}{I_0} = |Z|$.

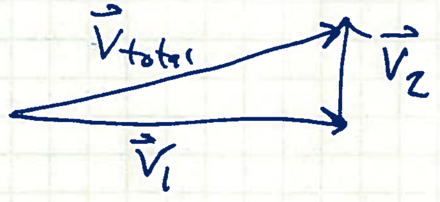
$$\vec{V}_L = \vec{I}_L \underbrace{(i\omega L)}$$

$\uparrow$ this means V_L <u>leads</u> I_L by 90°.

$$\vec{V}_C = \vec{I}_C \underbrace{\left(\frac{-i}{\omega C}\right)}$$

$\uparrow$ this means V_C <u>lags</u> I_C by 90°

<u>Phasor Diagrams</u> Show the geometric <u>relationships</u> betw. $\vec{V}$ & $\vec{I}$ for a circuit. This can be very useful when combined with a sum rule like $\vec{V}_{total} = \vec{V}_1 + \vec{V}_2$ or $\vec{I}_{total} = \vec{I}_1 + \vec{I}_2$



$$\therefore |V_{total}| = \sqrt{V_1^2 + V_2^2}$$

Normal Mode: In a multiparticle system, a normal mode is a type of motion where all particles oscillate at the same frequency.

- The number of normal modes is equal to the number of particles.
- Each normal mode goes at its own frequency.
- The general solution is a sum over normal modes:

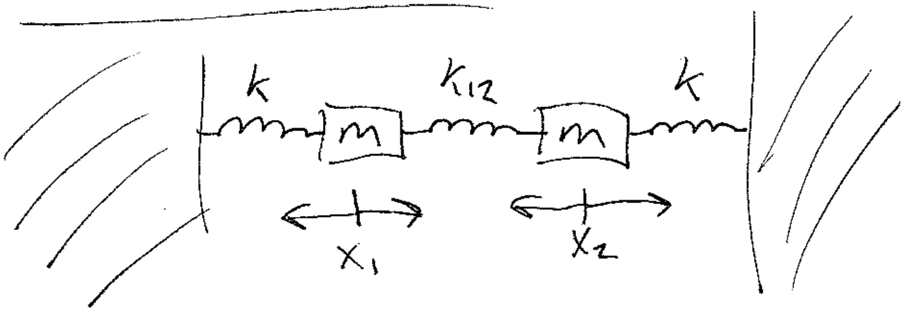
$$\vec{x}(t) = \sum_{n=1}^{2} c_n \vec{q}_n e^{i\omega_n t} \quad , \text{for a 2 particle system.}$$

The expansion coefficients $\{c_n\}$ are determined by the initial conditions and can be calculated using "Fourier's Trick":

$$c_n \equiv a_n + ib_n, \quad a_n = \frac{\vec{x}_0 \cdot \vec{q}_n}{|\vec{q}_n|^2} \quad , \quad b_n = -\frac{\vec{v}_0 \cdot \vec{q}_n}{\omega_n |\vec{q}_n|^2}$$

where $\vec{q}_n$ are the normal mode eigenvectors and $\vec{x}_0$ and $\vec{v}_0$ are the initial position and velocity.

2 coupled oscillators



Two normal modes:

$\vec{q}_1 = (1,1)$ = "symmetric mode"

$\vec{q}_2 = (1,-1)$ = "anti-symmetric mode"

Eqs. of Motion:

$$m\ddot{x}_1 + (k + k_{12})x_1 - k_{12}x_2 = 0$$

$$m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = 0$$

Solution:

$$(x_1(t), x_2(t)) = c_1(1,1)e^{i\omega_1 t} + c_2(1,-1)e^{i\omega_2 t}$$

For this system we called

$$\omega_1 = \omega_{small} = \sqrt{k/m}$$

$$\omega_2 = \omega_{Large} = \sqrt{\frac{k+2k_{12}}{m}}$$