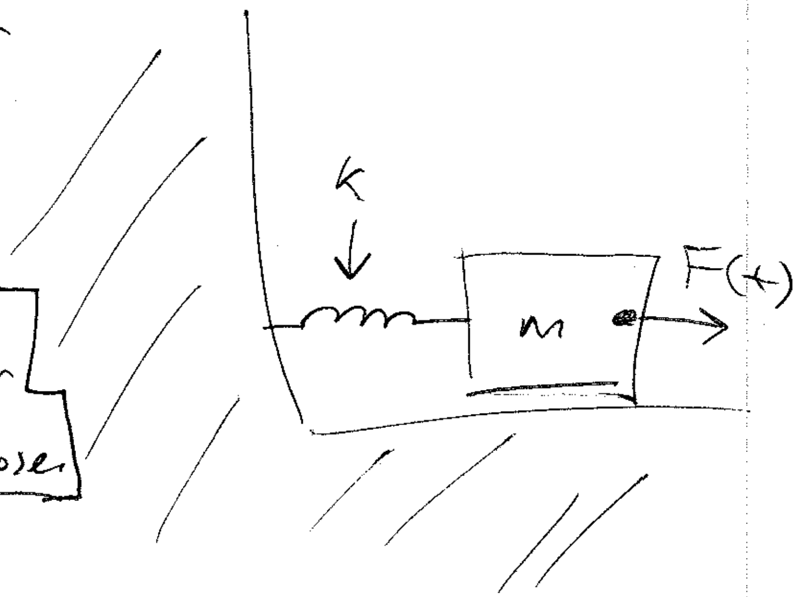
Recap of Forced Oscillator

Let $F(t) = F_0 e^{i\omega_F t}$

ω_F = forcing frequency, a parameter we can choose.



Eq. of Motion:

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega_F t}, \quad \text{where } \omega_0 = \sqrt{\frac{k}{m}} = \text{"natural frequency"}$$

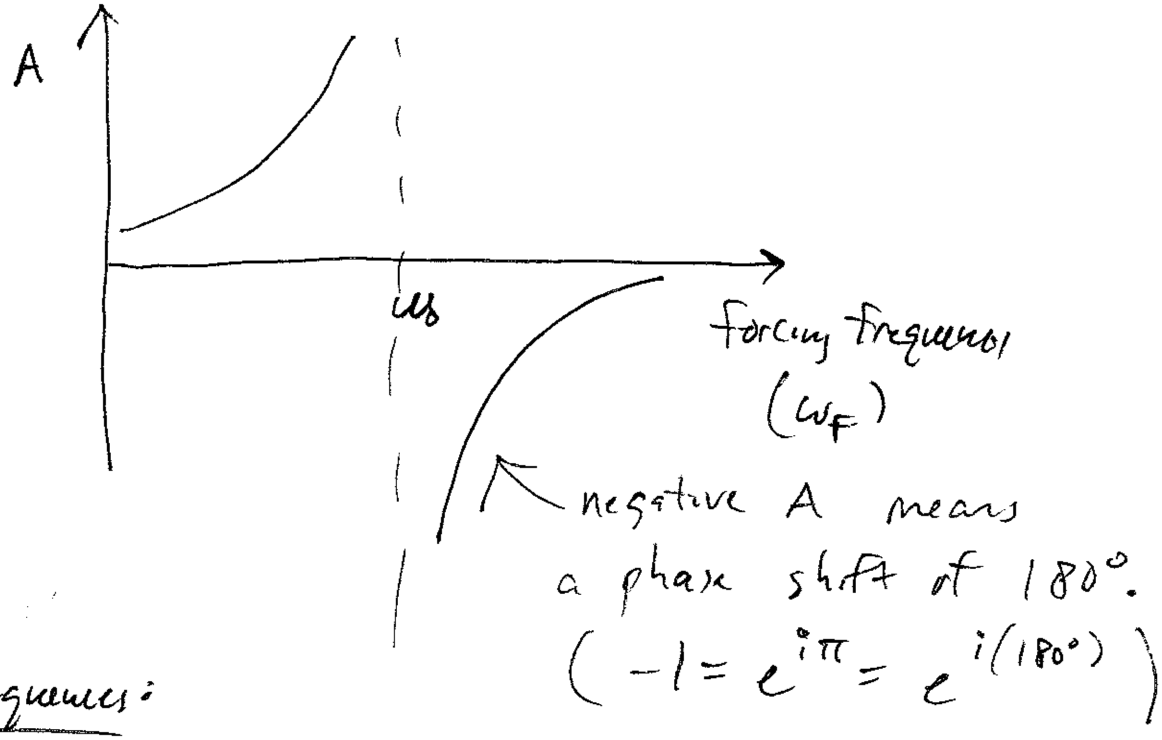
natural frequency

forcing frequency.

Solution:

$$\boxed{\begin{aligned} x(t) &= A e^{i(\omega_F t + \delta)} \\ \text{where } A &= \frac{F_0}{m(\omega_0^2 - \omega_F^2)} \\ \text{and } \delta &= \emptyset \end{aligned}}$$

A & δ are not free parameters, they are fixed. In particular, A depends on the forcing frequency:



Consequences:

(1) If $\omega_F \ll \omega_0$, then A is small and positive $\Rightarrow$ oscillator moves 100% in phase with the external force.

(2) If $\omega_F \gg \omega_0$, then A is small and negative $\Rightarrow$ oscillator moves 180° out-of-phase with the external force.

(3) If $\omega_F \approx \omega_0$, then A becomes very large $\Rightarrow$ Resonance.

If you force an oscillator to go at its natural frequency, the oscillator response will be large, which is a resonant phenomena.

Compare to the Simple Harmonic Oscillator (No Forcing)

- Always goes at ω_0.
- A & δ are free parameters which we can choose in order to satisfy the initial conditions.

For the Forced Oscillator, the initial conditions affect the transient (short-term) behavior. We will study this later. The solution we are studying now is the long-term behavior.

Simple Harmonic Oscillator with Damping.

Real mechanical oscillators always have some resistive force which is non-conservative. Resistive forces convert mechanical energy to heat. They must be modeled empirically. ~~We~~ A very simple model is:

$$F_{resistive} = -bv = -b\dot{x}$$

minus sign means the force acts opposite the velocity.

constant

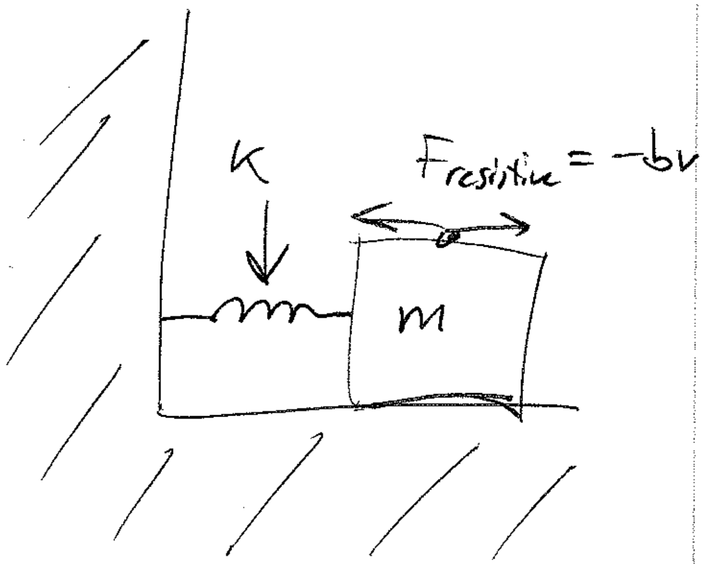
the oscillator's velocity

This model works reasonably well when v is not too ~~far~~ large

Eq. of Motion:

$$-kx - bv = m\ddot{x}$$

$v = \dot{x}$



$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \emptyset$$

no driving force

Define: $\gamma \equiv \frac{b}{m}$. Also, $\omega_0^2 \equiv \frac{k}{m}$. Then

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \emptyset}$$ Eq. of Motion.

$\Rightarrow$ small γ means very little drag.

$\Rightarrow$ large γ means large drag.

Guessed Solution:

$$x(t) = A e^{i(\omega_d t + \delta)}$$

ω_d = "frequency at which the damped oscillator goes" (not yet known!).

Also, A & δ are not known either.

Substitute the guess:

$$\ddot{x} = -\omega_d^2 x, \quad \dot{x} = i\omega_d x$$

Result: $\boxed{-\omega_d^2 x + i\omega_d \gamma x + \omega_0^2 x = 0}$

$x(t)$ divides out:

$$\boxed{-\omega_d^2 + i\omega_d \gamma + \omega_0^2 = 0}$$

A purely algebraic Equation for ω_d in terms of γ and ω_0.

Notice That

1. ω_d = purely real cannot satisfy this equation.
2. ω_d = purely imaginary also cannot satisfy this equation

We must allow ω_d to have both real & imaginary parts:

$$\omega_d = \underbrace{\omega_r}_{\text{real part}} + i\underbrace{\omega_i}_{\text{imaginary part}}$$

So our equation for ω_d says:

$$-(\omega_r + i\omega_i)^2 + i(\omega_r + i\omega_i)\gamma + \omega_0^2 = 0$$

Gathering real & imaginary terms we have

(1) $-\omega_r^2 + \omega_i^2 - \omega_i \gamma + \omega_0^2 = 0$ ← real terms must add to zero

(2) $i(-2\omega_r \omega_i + \omega_r \gamma) = 0$ ← imaginary terms must add to zero.

From 2 we must have: $\boxed{\omega_i = \frac{\gamma}{2}}$

Then substitute into 1:

$$\boxed{\omega_r^2 = \omega_0^2 - \frac{\gamma^2}{4}}$$

Therefore: $\omega_d = \left(\omega_0^2 - \frac{\gamma^2}{4}\right) + i\left(\frac{\gamma}{2}\right)$

So our guessed solution becomes:

$$x(t) = Ae^{i(\omega_d t + \delta)}$$

$$= Ae^{i((\omega_r + i\omega_i)t + \delta)}$$

$$= Ae^{-\omega_i t} e^{i(\omega_r t + \delta)}$$

$$\boxed{x(t) = Ae^{-\gamma t/2} e^{i(\omega_r t + \delta)}, \quad \text{where } \omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}$$

We still have 2 unknown parameters: A & δ.
These will be determined by initial conditions.

Comments

1 If we set $\gamma = 0$ (no damping), then we recover the SHO solution: $x(t) = Ae^{i(\omega_0 t + \delta)}$

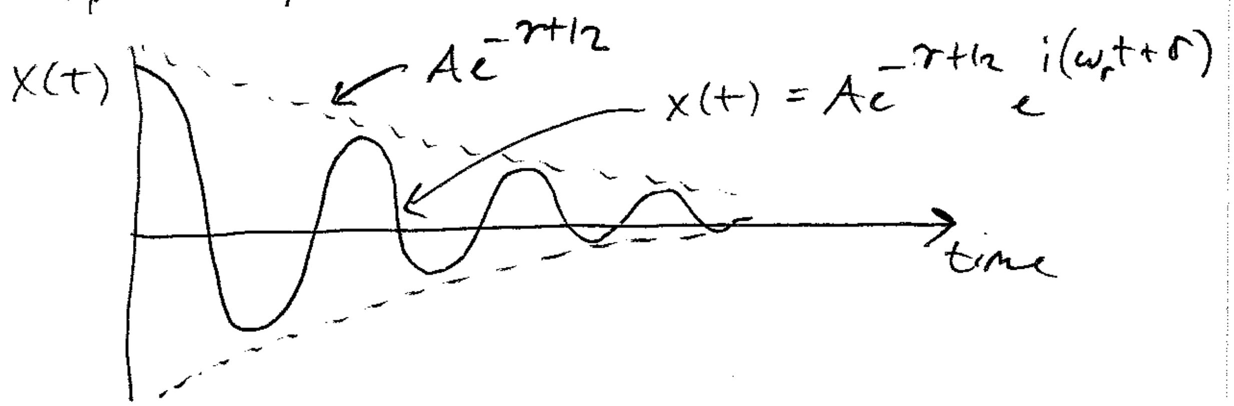
2) For non-zero γ, we have an oscillation at a frequency of

$$\omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

which is not the natural frequency due to the $\frac{\gamma^2}{4}$ term.

$\Rightarrow$ The damped oscillator goes at a frequency which is ~~slightly~~ smaller than the natural frequency. If the damping coefficient is small, then the damped oscillator frequency will only be very slightly less than ω_0.

3) The amplitude of the oscillation decays exponentially in time:



The mechanical energy of the oscillator is being converted into heat, and the amplitude is decreasing.

Energy Considerations

Mechanical energy = E ≡ Kinetic Energy + Potential Energy

$$E = KE + U$$

For a mass on a spring, $U = \frac{1}{2}kx^2$

$$KE = \frac{1}{2}m\dot{x}^2$$

$$\therefore\ E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

As a function of time,

$$x(t) = A\cos(\omega t + \delta)\ ,\ \text{and}\ \dot{x}(t) = -A\omega\sin(\omega t + \delta)$$

Also, $\omega_0^2 = k/m \implies k = m\omega_0^2$.

And in the case of <u>no</u> <u>damping</u>, $\underline{\omega = \omega_0}$.

$$\therefore\ E(t) = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \delta) + \frac{1}{2}m\omega_0^2A^2\cos^2(\omega_0 t + \delta)$$

$$= \frac{1}{2}m\omega_0^2A^2\left(\underbrace{\sin^2(\omega_0 t + \delta) + \cos^2(\omega_0 t + \delta)}_{1}\right)$$

$$\boxed{E(t) = \frac{1}{2}m\omega_0^2A^2 = \text{constant}\quad (\text{no damping})}$$

Energy is conserved.

But suppose the oscillator is "lightly damped":

Then $F_{resistive} = -\underset{\text{small}}{b}\dot{x}$ and $\gamma \equiv \frac{b}{m}$

Then $x(t) = \mathrm{Re}\left[Ae^{-\gamma t/2}e^{i(\omega_d t + \delta)}\right]$

$$= Ae^{-\gamma t/2}\cos(\omega_r t + \delta)$$

where $\cancel{\omega_d}\ \omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

We can write

$$x(t) = \underbrace{Ae^{-\gamma t/2}}_{A(t)} \cos(\omega_d t + \delta)$$

$x(t) = A(t)\cos(\omega_d t + \delta)$, and $\omega_d \approx \omega_0$ in the case of light damping.

Then the total mechanical energy is

$$E(t) = \tfrac{1}{2} m\omega_0^2 A^2 = \tfrac{1}{2} m\omega_0^2 \left(A(t)\right)^2$$
$$= \tfrac{1}{2} m\omega_0^2 \left(Ae^{-\gamma t/2}\right)^2$$

$$E(t) = \underbrace{\tfrac{1}{2} m\omega_0^2 A^2}_{\text{Call this } E_0} e^{-\gamma t}$$

For a lightly damped oscillator, E decays away exponentially with time constant $\left(\frac{1}{\gamma}\right)$. → $\boxed{E(t) = E_0 e^{-\gamma t}}$

(γ has units of $\frac{1}{\text{seconds}}$, so $\frac{1}{\gamma}$ has units of time)

"Q Factor" or "Quality Factor"

We want to define a quantity which tells us whether the oscillator loses energy quickly or slowly.

- "high Q" = "high quality" = low energy loss rate
- "low Q" = "low quality" = low energy loss rate

We define it this way:

Question: What fraction of the energy is lost in the time it takes for the oscillator's phase to advance by 1 radian?

Call this amount of time t_1. By definition

$$\underbrace{\omega_0 t_1}_{\substack{\text{phase} \\ \text{change} \\ \text{in time } t_1}} = 1 \implies \boxed{t_1 = \frac{1}{\omega_0}}$$

Time for phase to change by 1 radian

(Recall that $\cos(\underbrace{\omega_0 t}_{\substack{\text{phase} \\ \text{change}}} + \delta)$)

Answer: The fraction of energy which remains after time t is

$$\frac{E(t)}{E_0} = e^{-\gamma t} \approx 1 - \gamma t \quad \text{for small } \gamma \text{ and small } t.$$

$\therefore$ Fraction of energy lost is $\boxed{\gamma t}$.

Therefore in time $t_1 = \frac{1}{\omega_0}$, the fraction lost is

Fraction lost in time t_1. $\rightarrow \boxed{\gamma\left(\frac{1}{\omega_0}\right)}$

We define the "Q factor" to be the <u>inverse</u> of this fraction:

$$\frac{\gamma}{\omega_0} \equiv \frac{1}{Q}$$

"Quality factor" or "Q factor"

or $\boxed{Q \equiv \frac{\omega_0}{\gamma}}$ for a lightly damped oscillator.

<u>Ex</u>: If $Q = 100$, then the damped oscillator loses $\frac{1}{Q} = 1\%$ of its energy in time $t_1 = \frac{1}{\omega_0}$.

<u>Ex</u>: If $Q = 1000$, then the damped oscillator loses $\frac{1}{Q} = 0.1\%$ of its energy in time $t_1 = \frac{1}{\omega_0}$.

<u>Other ways to think about Q:</u>

$$Q \equiv \boxed{\frac{1}{\text{Fraction of energy lost in time } t = \frac{1}{\omega_0}}}$$

We can re-write it as

T = period = time for one complete cycle

$$T = \frac{1}{f} = \frac{2\pi}{\omega_0}$$

$\therefore$ Fraction of energy lost in one period $= \gamma T$

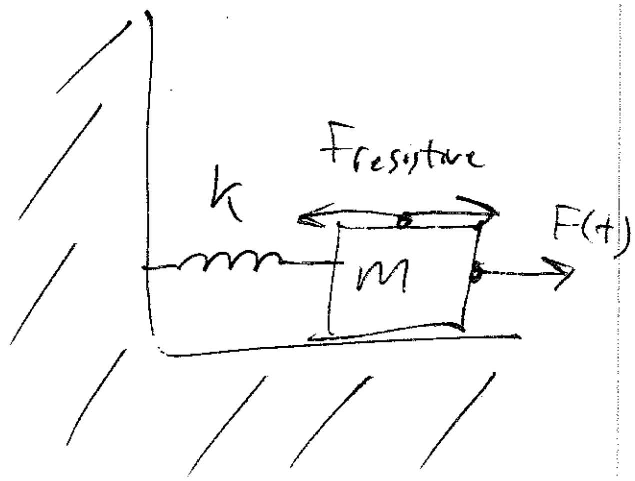
$$= \gamma\left(\frac{2\pi}{\omega_0}\right)$$

$$= \frac{2\pi}{(\omega_0/\gamma)}$$

$$\boxed{\text{Fraction of energy lost in one period} = \frac{2\pi}{Q}}$$

Forced Oscillator with damping

A harmonic oscillator with both damping and a driving force.



Eq. of Motion:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega_F t}$$

damping — natural frequency — forcing frequency

Guessed Solution: The forced oscillator with no damping oscillates at the forcing frequency. Let's try the same solution for the damped case:

$$x(t) = A e^{i(\omega_F t + \delta)} \leftarrow \text{Guessed Solution}$$

forcing frequency

The usual question is: what are A & δ?

substitute: $\ddot{x} = -\omega_f^2 x$, $\dot{x} = i\omega_f x$

$$\Rightarrow -\omega_f^2 x + i\omega_f \gamma x + \omega_0^2 x = \frac{F_0}{m} e^{i\omega_f t}$$

$x = A e^{i(\omega_f t + \delta)}$

So $e^{i\omega_f t}$ cancels everywhere:

$$\left[-\omega_f^2 + i\omega_f \gamma + \omega_0^2\right] A e^{i\delta} = \frac{F_0}{m}$$

$$\underbrace{A(\omega_0^2 - \omega_f^2) + i(\omega_f \gamma) A}_{\text{A complex number in cartesian form}} = \underbrace{\frac{F_0}{m} e^{-i\delta}}_{\text{A complex number in polar form.}}$$

We have a real eq. and an imaginary eq.:

$$\boxed{A(\omega_0^2 - \omega_f^2) = \frac{F_0}{m}\cos(-\delta) = \frac{F_0}{m}\cos\delta} \quad (1) \text{ Real Eq.}$$

and $$\boxed{A\omega_f \gamma = \frac{F_0}{m}\sin(-\delta) = -\frac{F_0}{m}\sin\delta} \quad (2) \text{ Imag. Eq.}$$

Take the ratio of the two equations to eliminate A:

$$\frac{(2)}{(1)}: \quad \frac{\omega_f \gamma}{(\omega_0^2 - \omega_f^2)} = \frac{-\sin\delta}{\cos\delta} = -\tan\delta$$

δ = phase shift of $x(t)$ relative to the forcing function $= \tan^{-1}\left[\frac{-\omega_F \gamma}{\omega_0^2 - \omega_F^2}\right]$

Phase Shift of the forced oscillator with damping :

$$\boxed{\delta(\omega_F) = -\tan^{-1}\left[\frac{\omega_F \gamma}{\omega_0^2 - \omega_F^2}\right]}$$

phase shift depends on the forcing frequency.

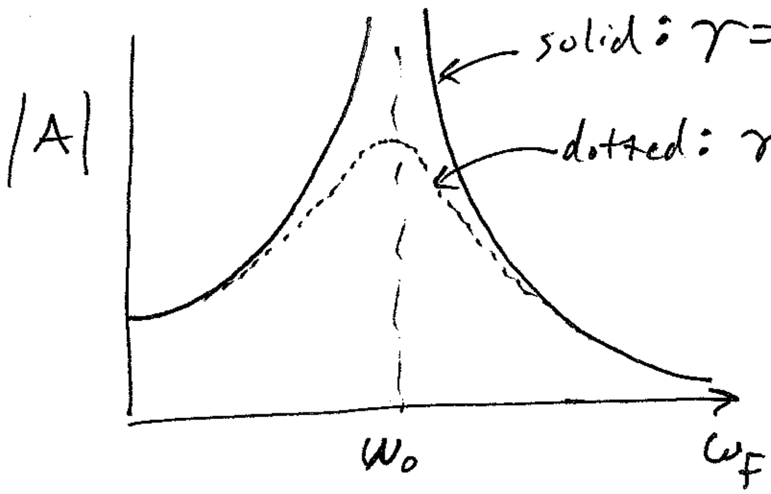
Also, we can solve for A by squaring 1 & 2 and adding:

$$1^2 + 2^2 \Rightarrow A^2\left[\left(\omega_0^2 - \omega_F^2\right)^2 + \left(\omega_F \gamma\right)^2\right] = \left(\frac{F_0}{m}\right)^2 \left[\underbrace{\cos^2\delta + \sin^2\delta}_{1}\right]$$

Amplitude of the forced Oscillator with damping, as a function of the forcing frequency.

$$\boxed{A(\omega_F) = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega_F^2\right)^2 + \left(\omega_F \gamma\right)^2}}}$$

Just like the forced oscillator with no damping, we have a resonance where A becomes large when ω_F is chosen to be near ω_0:

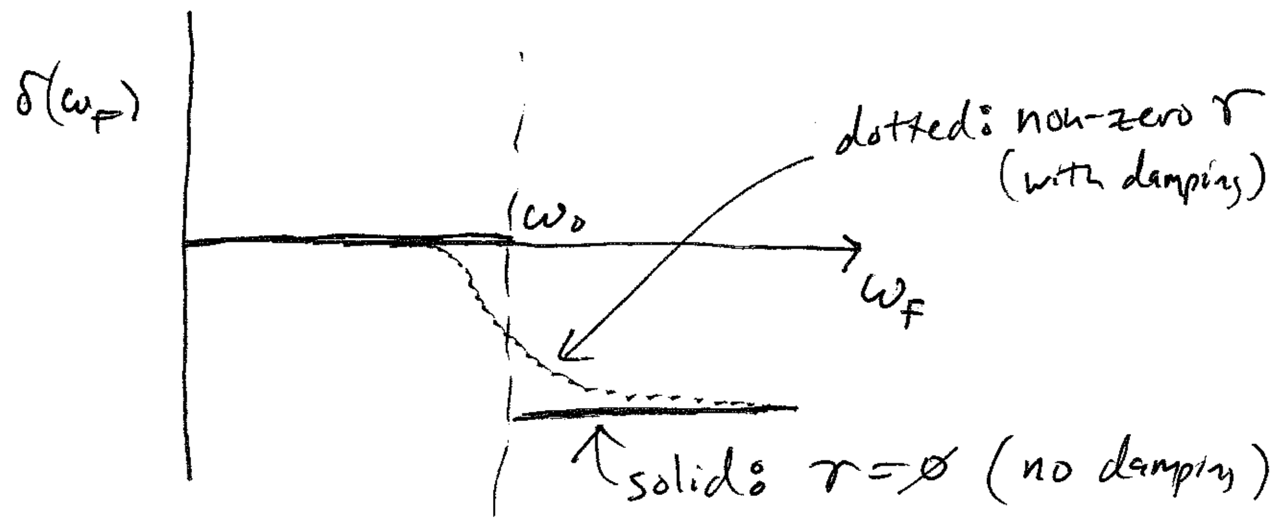


The damping coefficient γ removes the infinite amplitude that we had when $\omega_F = \omega_0$ for the un-damped forced oscillator.

With damping, the amplitude A has a maximum when $\omega_F \approx \omega_0$, but the amplitude remains finite.

Phase Shift: Recall that δ is the phase shift of the oscillator relative to the driving force. (we define $t = \emptyset$ so that the driving force phase is exactly zero.)

Plot the phase shift function:



The phase shift is <u>negative</u>: the oscillator <u>lags behind</u> the forcing function. When $\omega_F \approx \omega_0$, the phase shift is about -90°.

<u>Transient Behavior</u>

We have been studying the <u>long-term</u> behavior of the forced oscillator: the motion of the oscillator once it settles into a simple repeating pattern. The long-term behavior <u>does not depend on the initial conditions</u>. So there are <u>no free parameters</u> in the long-term solution.

The initial conditions <u>will affect</u> the <u>short-term behavior</u>, however. This is called the <u>transient</u>

solution. Any system with non-zero damping will have its transient behavior die out as time goes forward, leaving only the long term solution. However, if the damping coefficient is small, then it may take a long time for the transient solution to go away.

How can we study the transient behavior?

The trick is to make the following observation: Consider a forced oscillator with damping, and seperately a damped oscillator with no forcing.

Forced Oscillator w/ damping	Damped Oscillator
Eq. of Motion: $\ddot{x}_F + \gamma \dot{x}_F + \omega_0^2 x_F = \frac{F_0}{m} e^{i\omega_F t}$	$\ddot{x}_d + \gamma \dot{x}_d + \omega_0^2 x_d = 0$
Solution: $\underbrace{x_F(t)}_{\text{forced oscillator solution}} = A(\omega_F) e^{i(\omega_F t + \underbrace{\delta_F}_{\text{forced oscillator phase shift}})}$	$x_d(t) = B e^{-\gamma t/2} e^{i(\underbrace{\omega_d}_{\text{damped frequency}} t + \underbrace{\delta_d}_{\text{damped phase shift}})}$
Number of free parameters: Zero.	Two: B and δ_d

Notice that if we add the forced solution to the damped solution, we get a new solution which also satisfies the forced equation of motion:

General Solution of forced oscillator with damping:

$$x(t) = \underbrace{x_F(t)}_{\text{forced solution}} + \underbrace{x_d(t)}_{\text{damped solution}}$$

$$= A(\omega_F)e^{i(\omega_F t+\delta_F)} + Be^{-\gamma t/2}e^{i(\omega_d t+\delta_d)}$$

two free parameters (B & δ_d).

Let's show that this works:

Eq. of Motion: $\ddot{x} + \gamma\dot{x} + \omega_0^2 x \stackrel{?}{=} \frac{F_0}{m}e^{i\omega_F t}$

$$(\ddot{x}_F + \ddot{x}_d) + \gamma(\dot{x}_F + \dot{x}_d) + \omega_0^2(x_F + x_d) \stackrel{?}{=} \frac{F_0}{m}e^{i\omega_F t}$$

Gather together the damped terms:

$$\underbrace{(\ddot{x}_d + \gamma\dot{x}_d + \omega_0^2 x_d)} + (\ddot{x}_F + \gamma\dot{x}_F + \omega_0^2 x_F) \stackrel{?}{=} \frac{F_0}{m}e^{i\omega_F t}$$

This equals zero according to the x_d equation of motion.

$$\ddot{x}_F + \gamma \dot{x}_F + \omega_0^2 x_F \overset{?}{=} \frac{F_0}{m} e^{i\omega_F t}$$

Is this true? Yes ✓, because this is the forced Eq. of Motion, and x_F is its solution.

So our general solution is:

$$\boxed{x(t) = A(\omega_F) e^{i(\omega_F t + \delta_F)} + B e^{-\gamma t/2} e^{i(\omega_d t + \delta_d)}}$$

B and δ_d should be chosen so that the initial conditions are satisfied.

Notice that the second term in the solution dies out exponentially as time goes forward due to the $e^{-\gamma t/2}$ ~~term~~ factor. This is the transient ~~soln~~ part of the solution.

After the transient dies out, we are left with the long-term behavior described by the first term in the solution.

Simple Example. Suppose we start with

$x(t=0) = 0$ and $\dot{x}(0) = 0$,

and suppose we drive the oscillator at the resonant frequency: $\omega_F = \omega_0$.

Also, to simplify things further, assume that damping is very small, so that

$$\omega_d = \sqrt{\omega_0^2 - \frac{\underbrace{\gamma^2}_{\text{Assume small}}}{4}} \approx \omega_0$$

This is the "high-Q" assumption

Then the solution is

$$\boxed{A(\omega_f) = A(\omega_0) = \frac{F_0}{m\omega_0\gamma}}$$

$$\tan(\delta_f(\omega_f)) = \frac{-\omega_0\gamma}{\omega_0^2-\omega_0^2} \rightarrow -\infty, \text{ so } \boxed{\delta_F(\omega_0) = -\pi/2}$$

Then the real part of the solution is:

$$x(t) = \frac{F_0}{m\omega_0\gamma}\underbrace{\cos(\omega_0 t - \pi/2)}_{\sin(\omega_0 t)} + Be^{-\gamma t/2}\cos(\omega_0 t + \delta_d)$$

$$x(t) = \frac{F_0}{m\omega_0\gamma}\sin(\omega_0 t) + Be^{-\gamma t/2}\cos(\omega_0 t + \delta_d)$$

Now determine B & δ_d from the initial conditions:

$$x(t=0) = 0 = B e^{-\gamma t/2} \cos(\delta_d) \Rightarrow \boxed{\delta_d = \pi/2}$$

And

$$\dot{x}(t=0) = 0 = \frac{F_0}{m\gamma}\cos(\omega_0 t) + B\Big(-\omega_0 \sin\big(\omega_0 t + \overset{\pi/2}{\delta_d}\big)\Big) e^{-\gamma t/2}$$

$$- \frac{B\gamma}{2}\Big(\cos\big(\omega_0 t + \underset{\pi/2}{\delta_d}\big)\Big) e^{-\gamma t/2}$$

At $t=0$:

$$0 = \frac{F_0}{m\gamma} - B\omega_0 \underbrace{\sin(\pi/2)}_{1} - \frac{B\gamma}{2}\underbrace{\cos(\pi/2)}_{0}$$

$$0 = \frac{F_0}{m\gamma} - B\omega_0 \Rightarrow \boxed{B = \frac{F_0}{m\omega_0\gamma}}$$

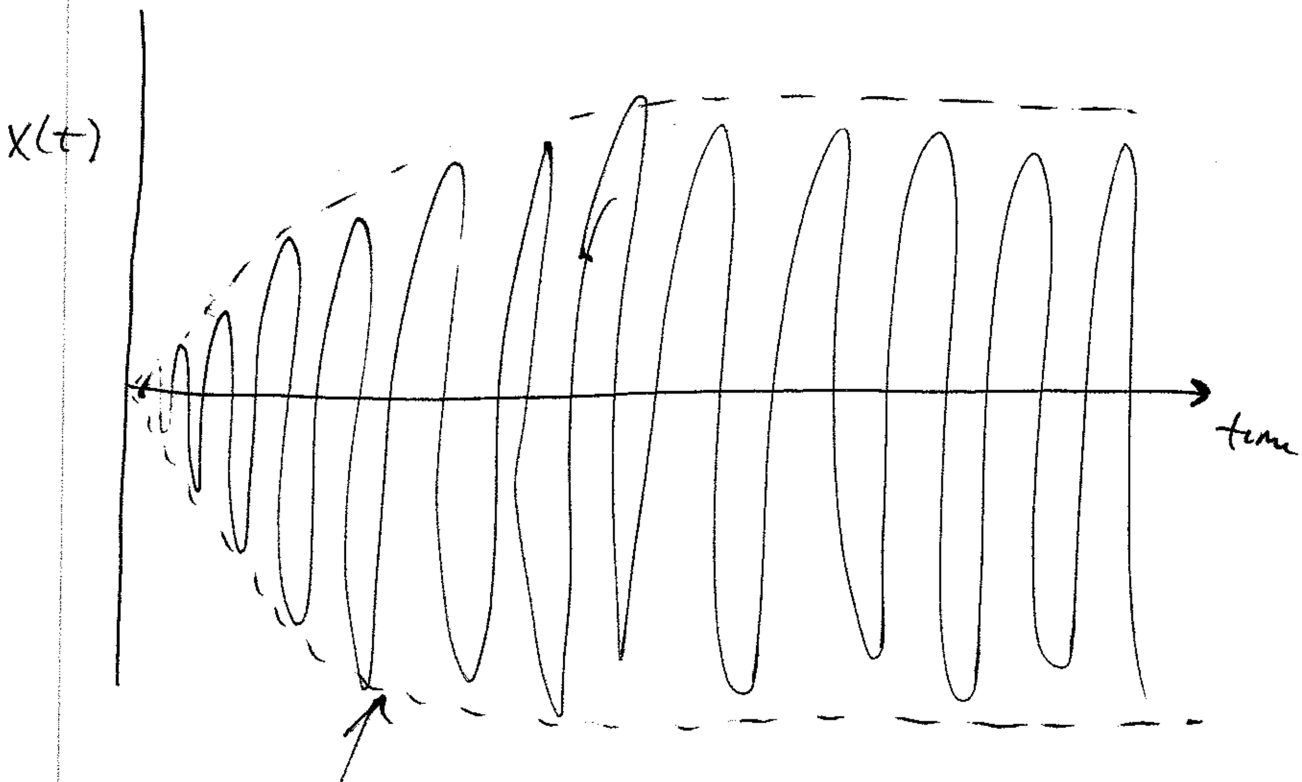
Finally, when $\omega_F = \omega_0$, $\omega_d \approx \omega_0$, and $x(0) = 0$,
and $\dot{x}(0) = 0$,

Then

$$x(t) = \frac{F_0}{m\omega_0\gamma}\sin(\omega_0 t) + \frac{F_0}{m\omega_0\gamma} e^{-\gamma t/2} \underbrace{\cos\left(\omega_0 t + \frac{\pi}{2}\right)}_{-\sin(\omega_0 t)}$$

$$\boxed{x(t) = \frac{F_0}{m\omega_0\gamma}\left(1 - e^{-\gamma t/2}\right)\sin(\omega_0 t)}$$

What does it look like:



exponential factor: $\left(1 - e^{-\gamma t/2}\right)$