Sound Waves

Let Pressure be described by

$$P(x,t) = P_0 + P'(x,t)$$

P_0: average, equilibrium pressure

$P'(x,t)$: deviation from equilibrium

And density be:

$$\rho(x,t) = \rho_0 + \rho'(x,t)$$

ρ_0: average

$\rho'(x,t)$: deviation

Then it can be shown that

$$\boxed{\frac{\partial^2 p'}{\partial x^2} = \frac{1}{V_s^2}\frac{\partial^2 p'}{\partial t^2}}$$ Classical Wave Eq.

where $V_s = \sqrt{\dfrac{\gamma P_0}{\rho_0}}$ = velocity of sound

For air, $\rho_0 = 1.2 \dfrac{g}{cm^3} = 1.2 \dfrac{kg}{m^3}$

$$P_0 = 1 \text{ atm} = 1.01 \times 10^5 \text{ Pa} = 1.01 \times 10^5 \frac{N}{m^2}$$

and $$\gamma = \frac{\text{heat capacity at constant pressure}}{\text{heat capacity at constant volume}}$$

$$= 1.4 \text{ for air}$$

so $v_s \approx 343$ m/s

Acoustic Impedance per unit area $= \rho_0 v_s = \sqrt{\gamma \rho_0 P_0} \approx 413 \frac{N \cdot s}{m^3}$

$$= \frac{\text{ratio of sound pressure}}{\text{to particle velocity.}}$$

The acoustic impedance of an acoustic component (such as a loud speaker or ~~the~~ an organ pipe) is the ratio of sound pressure to particle velocity at the connection point.

Normal Modes of open & closed pipes

A pipe or tube can act as a simple musical instrument, with the boundary conditions determining the allowed normal modes (harmonic series).

The rules are that

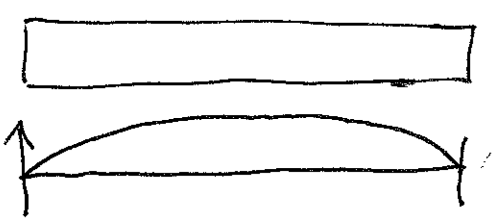
1. A closed end forces the displacement of the air molecules to be zero, but the pressure change can be non-zero.

2. An open end forces the pressure change to be zero, but the displacement of air molecules can be non-zero.

	molecule displacement	pressure change
open end	non-zero (anti-node)	zero (node)
closed end	zero (node)	non-zero (anti-node)

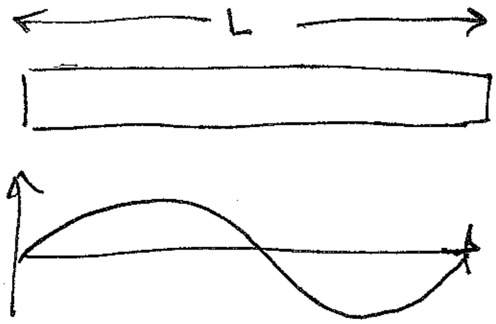
For a tube closed at both ends, the fundamental looks like:

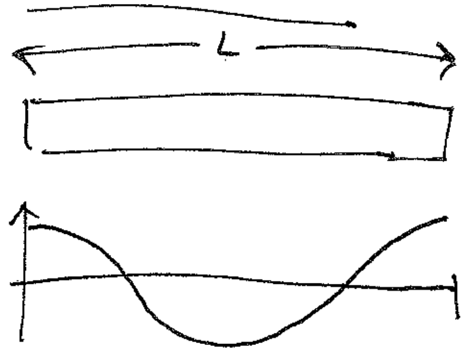
molecule displacement



Pressure change

The 1^{st} harmonic (2^{nd} excited state)





So in this case the allowed wavelengths are multiples of ~~$\lambda/2$~~.

$$L = n\left(\frac{\lambda}{2}\right), \quad n = 1, 2, 3, 4$$

$$\lambda_n = \frac{2L}{n}$$

The associated frequencies can be determined by requiring that (wavelength) × (frequency) = speed of sound:

$$\lambda_n f_n = v_s$$

$$f_n = \frac{v_s}{\lambda_n} = \frac{n v_s}{2L}$$

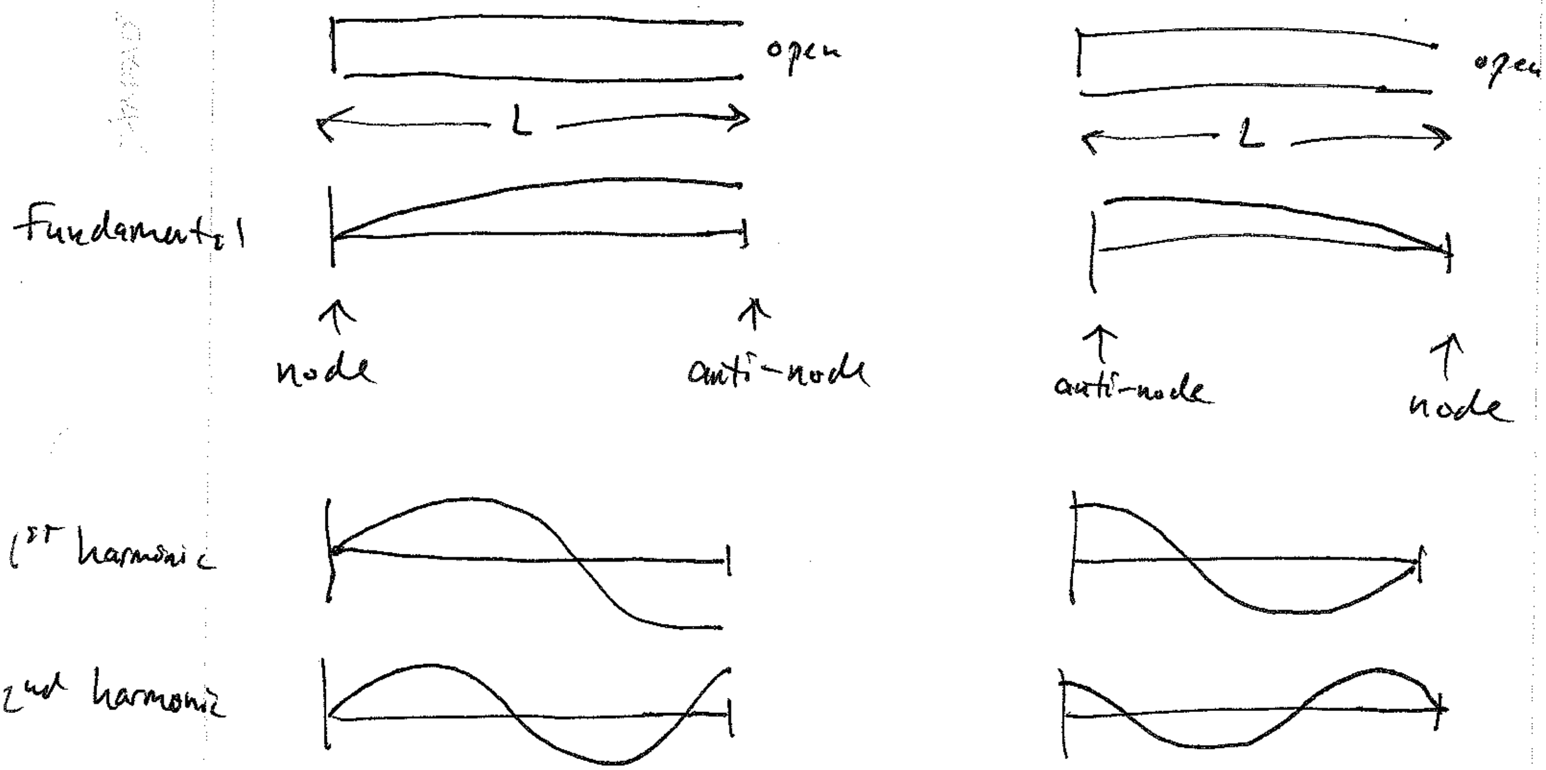
Angular frequencies then are:

$$\omega_n = 2\pi f_n = \frac{n\pi v_s}{L}.$$

But a tube closed at both ends would make sounds that are difficult to hear. So consider one end open, like a trumpet, or clarinet:

molecule displacement

Pressure Change



The pattern is an odd number of 1/4 wavelengths.

$$L = n\left(\frac{\lambda}{4}\right), \quad \text{odd } (n) \text{ only.}$$

$$\lambda_n = \frac{4L}{n}, \quad \text{odd } (n) \text{ only.}$$

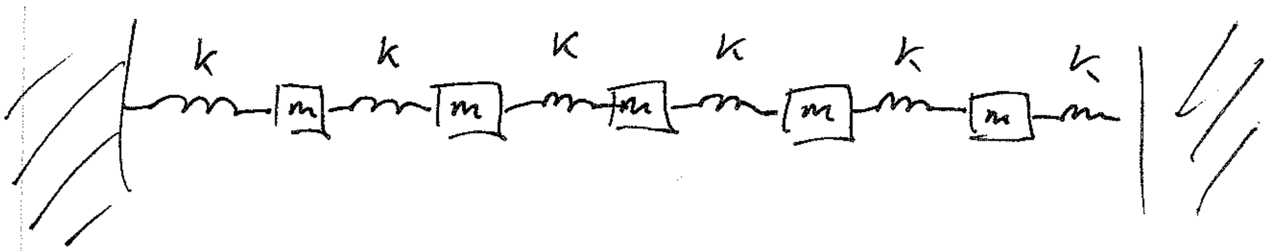
$$f_n = \frac{v_s}{\lambda_n} = \frac{n v_s}{4L}, \quad n = 1, 3, 5$$

If the tube is a clarinet, we can effectively shorten the length by opening a key in the middle. This forces the pressure change to be zero at the location of the key, creating a node in the pressure change and an anti-node in the molecular displacement.

A tube which is open at both ends, like some organ pipes, will have a different set of normal modes, again determined by the boundary conditions.

More on longitudinal Oscillations: Elastic Modulus

Consider again masses connected by springs:



Assume that all springs are identical and all masses identical.

Eq. of Motion for particle # p:

$$m\ddot{x}_p = k(x_{p+1} - x_p) - k(x_p - x_{p-1})$$

$$\uparrow \qquad\qquad \uparrow$$

$$(m\omega_0^2) \qquad (m\omega_0^2)$$

$$\boxed{\ddot{x}_p + 2\omega_0^2 x_p - \omega_0^2(x_{p+1} + x_{p-1}) = \emptyset}$$

This Eq. of Motion is identical to that of the loaded string which moves in the transverse direction, However in this system the motion is along the direction of the springs (longitudinal).

We can take the continuum limit to get the Classical Wave Equation. First, let's put the (k's) back in the equation; and let's use ξ (ksi) to represent displacement (instead of x):

$$m\frac{d^2\xi(x)}{dt^2} = k\Big[\big(\xi(x+\Delta x) - \xi(x)\big) - \big(\xi(x) - \xi(x-\Delta x)\big)\Big]$$

Divide by ΔX on both sides:

$$\frac{m}{\Delta x}\frac{d^2\xi(x)}{dt^2} = k\left[\frac{\left(\xi(x+\Delta x)-\xi(x)\right)}{\Delta x} - \frac{\left(\xi(x)-\xi(x-\Delta x)\right)}{\Delta x}\right]$$

Multiply & divide by Δx again on RHS:

$$\underbrace{\frac{m}{\Delta x}}_{\rho\,=\,\text{mass density}}\frac{d^2\xi(x)}{dt^2} = (k\,\Delta x)\left[\frac{\frac{\left(\xi(x+\Delta x)-\xi(x)\right)}{\Delta x} - \frac{\left(\xi(x)-\xi(x-\Delta x)\right)}{\Delta x}}{\Delta x}\right]$$

In the limit where $\Delta x \to 0$, the RHS becomes the 2^{nd} spatial derivative: $\dfrac{d^2\xi(x)}{dx^2}$.

The constant $(k\,\Delta x)$ is a property of the material called the "elastic modulus" or sometimes "Young's Modulus". It has units of

$$k\,\Delta x = \frac{N}{m}\cdot m = \text{Newton} = \text{Force}.$$

We use the symbol $E \equiv k\,\Delta x$ = elastic modulus.

So the wave equation for longitudinal oscillations in the continuum limit appears as (using partial derivatives now)

$$\boxed{\frac{\partial^2 \xi(x,t)}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \xi(x,t)}{\partial x^2}}$$

And the phase velocity is

or $$\boxed{\frac{\partial^2 \xi(x,t)}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 \xi(x,t)}{\partial t^2}}$$

$$v_p = \sqrt{\frac{E}{\rho}}$$

~~For steel, $E \approx 2 \times 10^{11}$ N~~

If our material is a 3-dimensional Rectangular block, then we should let the density ρ be the mass per unit volume, rather than mass per unit distance $\left(\frac{kg}{m^3}\right.$ instead of $\left.\frac{kg}{m}\right)$.

Then the units of E should be $\left(\frac{N}{m^2}\right)$ rather than simply (N).

$$E = \text{Elastic Modulus} = \frac{N}{m^2} \text{ in 3 dimensions.}$$

For steel, $E = 2 \times 10^{11} \frac{N}{m^2}$ and $\rho = 7.75 \times 10^3 \frac{kg}{m^3}$,

so the speed of sound is $v_s = v_p = \sqrt{\frac{E}{\rho}} = 5 \times 10^3 \frac{m}{s}$

Classical Wave Equation.

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

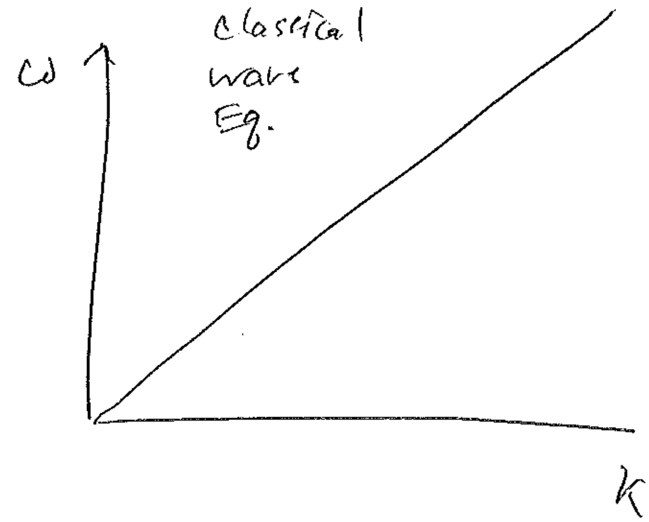
or $$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$$

Guess $\psi(x,t) = Ae^{i(kx-\omega t)}$

$$-\omega^2 = v^2(-k^2)$$

$$\omega = vk$$

$$\omega(k) = vk$$



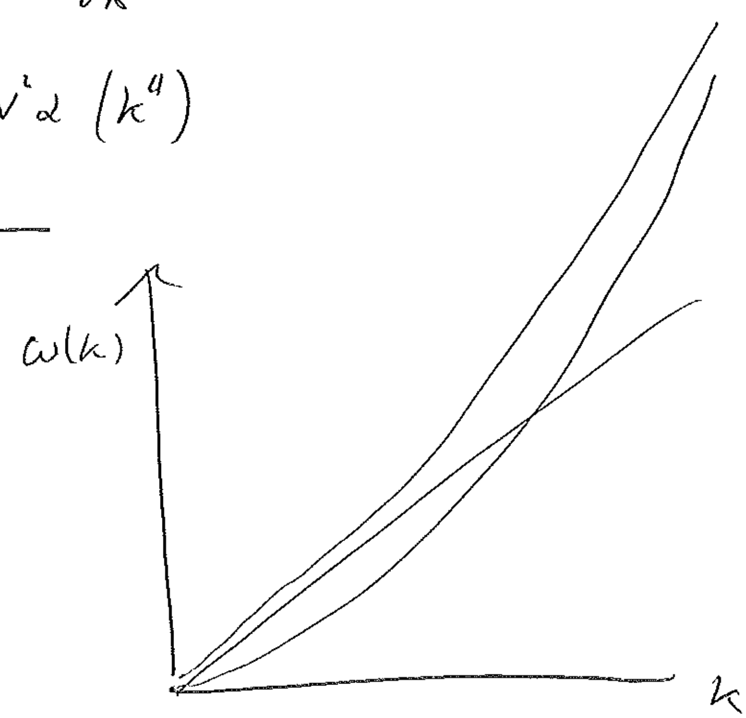
"Stiff steel piano wire" Equation

$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2} - v^2\alpha\frac{\partial^4\psi}{\partial x^4}$$

Guess $\psi(x,t) = Ae^{i(kx-\omega t)}$

$$-\omega^2 = v^2(-k^2) - v^2\alpha(k^4)$$

$$\omega(k) = \omega = vk\sqrt{1+\alpha k^2}$$



Free Quantum Particle

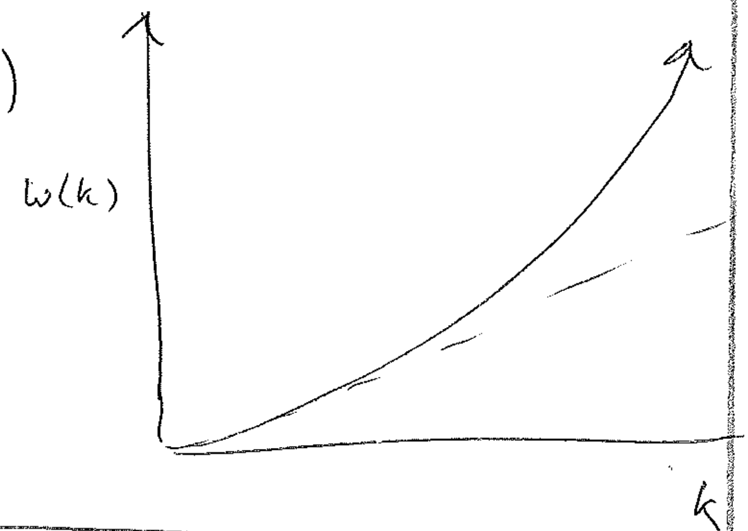
$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$$

Guess $\psi(x,t) = Ae^{i(kx-\omega t)}$

$$i\hbar(-i\omega) = \frac{-\hbar^2}{2m}(-k^2)$$

$$\omega = \frac{\hbar k^2}{2m}$$

$$\omega(k) = \frac{\hbar k^2}{2m}$$



Water waves:

Small ripple water waves:

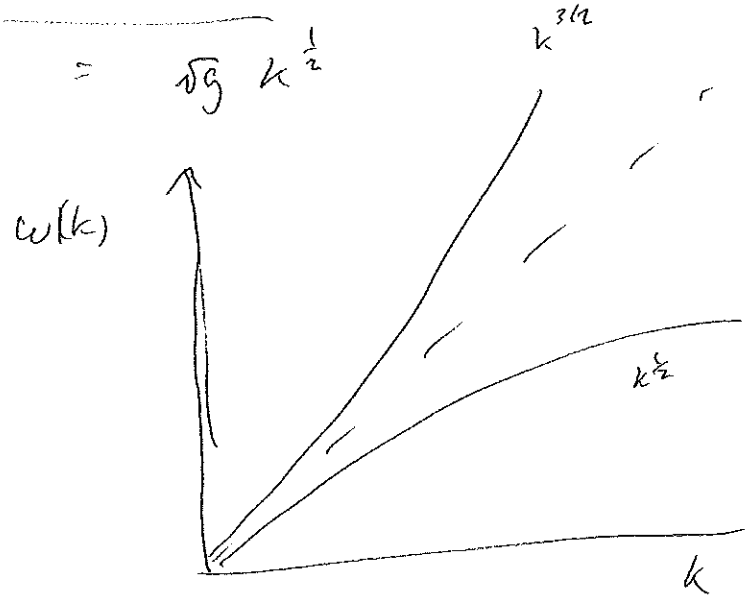
$$\omega(k) = \sqrt{\frac{\sigma k^3}{\rho}} = \sqrt{\frac{\sigma}{\rho}}\, k^{3/2}$$

σ = surface tension

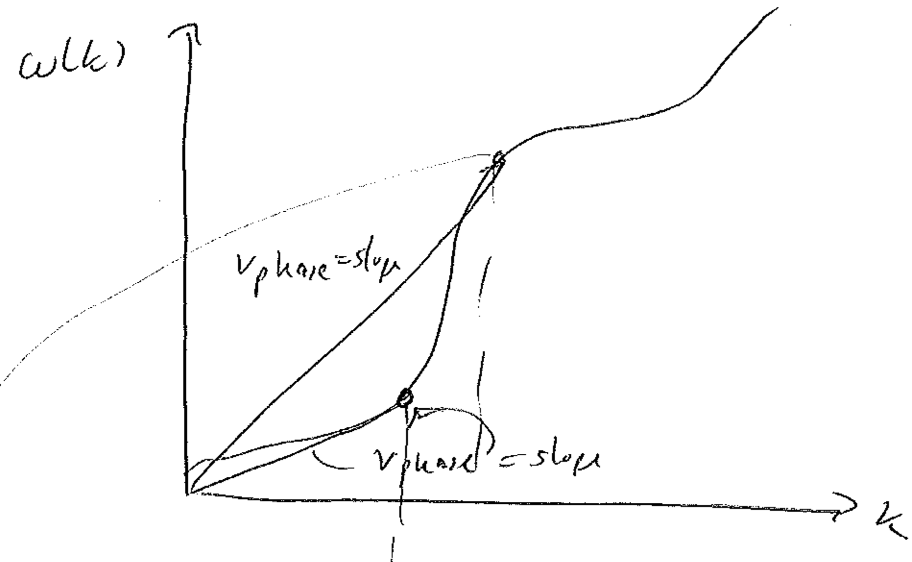
ρ = density

Long Wavelength in deep water

$$\omega(k) = \sqrt{gk} = \sqrt{g}\, k^{\frac{1}{2}}$$



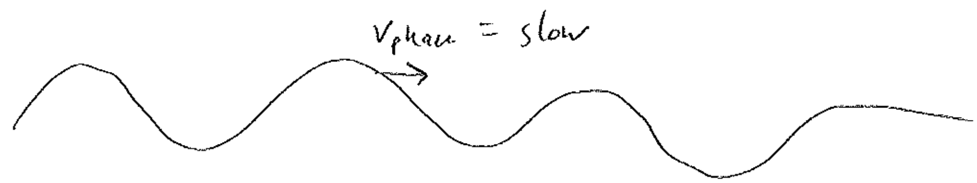
In general for any harmonic wave the phase velocity is the ratio of $\frac{\omega}{k}$:

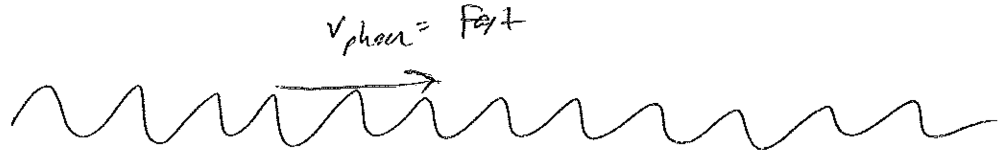


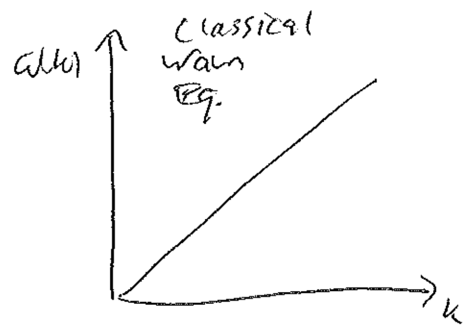
$$\psi(x,t) = A e^{i(kx-\omega t)} = A e^{ik(x-\frac{\omega}{k}t)}$$

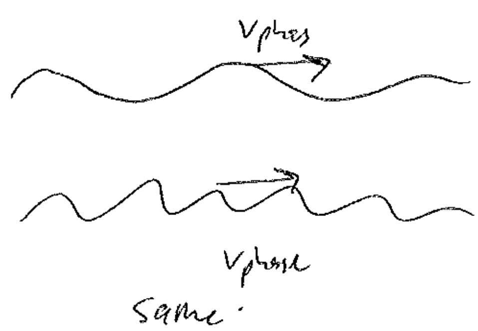
$$= A e^{ik(x - v_{phase} t)}$$

$$v_{phase} \equiv \frac{\omega}{k}$$









For any ~~sta~~ system, the relationship between ω and k is called the "<u>dispersion relation</u>"

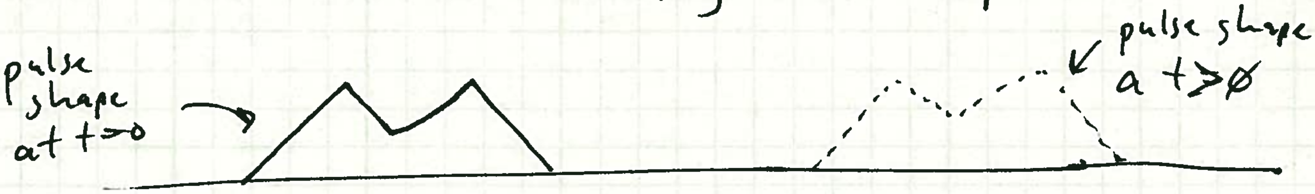
The classical wave equation has a <u>linear</u> dispersion relation:

$$\boxed{\omega(k) = v_p k}$$ ← linear dispersion relation (classical wave equation)

whereas the loaded string has a non-linear dispersion relation:

$$\boxed{\omega(k_n) = 2\omega_0 \sin\left(\frac{k_n l}{2}\right)}$$ ← non-linear dispersion relation (loaded string)

A system which has a <u>linear</u> dispersion relation has a special property: A propagating pulse will travel without changing its shape:



We can show this as follows. The pulse can be described as a sum over normal modes. But the normal modes are continuous, so the sum over normal modes is a Fourier Transform:

$$y(x,t) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \underbrace{A(k)}_{\text{its coefficient}}\, \underbrace{e^{i(kx)}}_{\text{normal mode}}\, \underbrace{e^{-i\omega t}}_{\text{its frequency}}\right]$$

sum over

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) \, e^{i(kx - \omega t)}$$

Now suppose that the system has a <u>linear dispersion</u> relation:

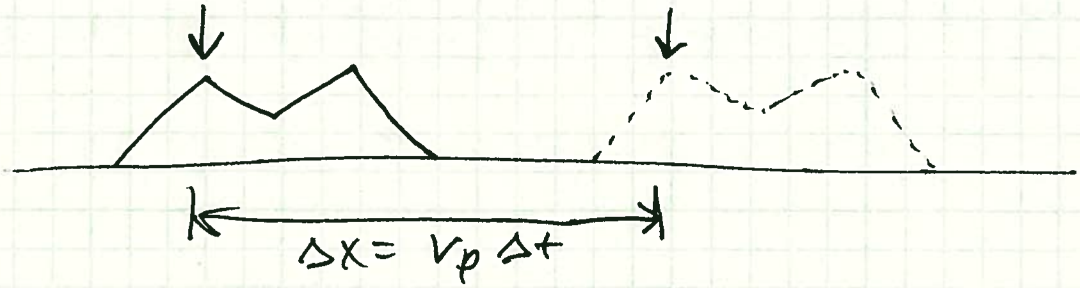
$$\omega = v_p k$$

Then we have

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) \, e^{i(kx - (v_p k)t)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, A(k) \, e^{ik\underbrace{(x - v_p t)}}$$

This says that as time goes forward, if we keep advancing x at speed v_p, then the value of y will stay the same. So the shape of the pulse does not change



Therefore, if $\omega = v_p k$ for the system, then pulses <u>do not disperse</u>. They <u>maintain</u> <u>their shape</u>.

$$\boxed{\text{A linear dispersion relation means that pulses do not disperse.}}$$

This is a special case behavior for systems with linear dispersion relations. But suppose that we have a non-linear dispersion relation. For example, suppose

$\omega \sim k^2$. This happens in quantum mechanics when describing a free particle. In that case,

$$\omega = \frac{\hbar k^2}{2m}.$$

How does a pulse propagate in a system like this?

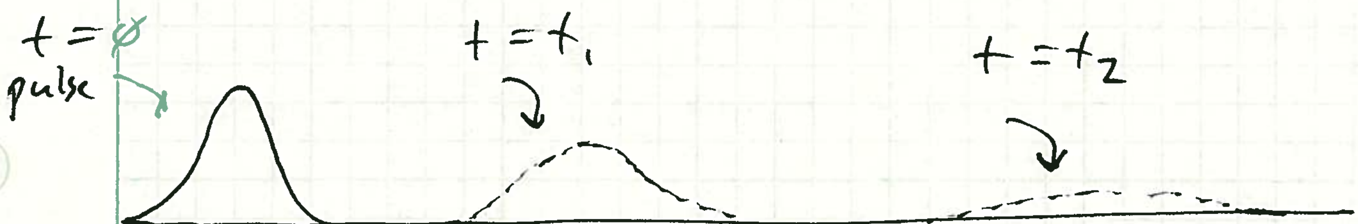
$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; A(k) e^{ikx} e^{-i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{ikx} e^{-i\left(\frac{\hbar k^2}{2m}\right)t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{ik\underbrace{\left(x - \frac{\hbar k}{2m} t\right)}}$$

Here, as time goes forward, we need to advance x at a speed of $v_p = \frac{\hbar k}{2m}$ to keep the argument of the exponential the same.

But the speed $v_p = \frac{\hbar k}{2m}$ is different for every ~~normal~~ wavenumber (k). That is, each normal mode ~~as~~ advances at its own velocity, they do not advance together. This means that the various normal modes will disperse; with some travelling fast, and some travelling slowly, and the pulse will dissappear.



Therefore the classical wave equation describes systems which have no dispersion. In these systems, a pulse can travel forever. An example of this in nature is electromagnetic waves in vacuum, (or waves on an ideal string.)

Information transmission and group velocity

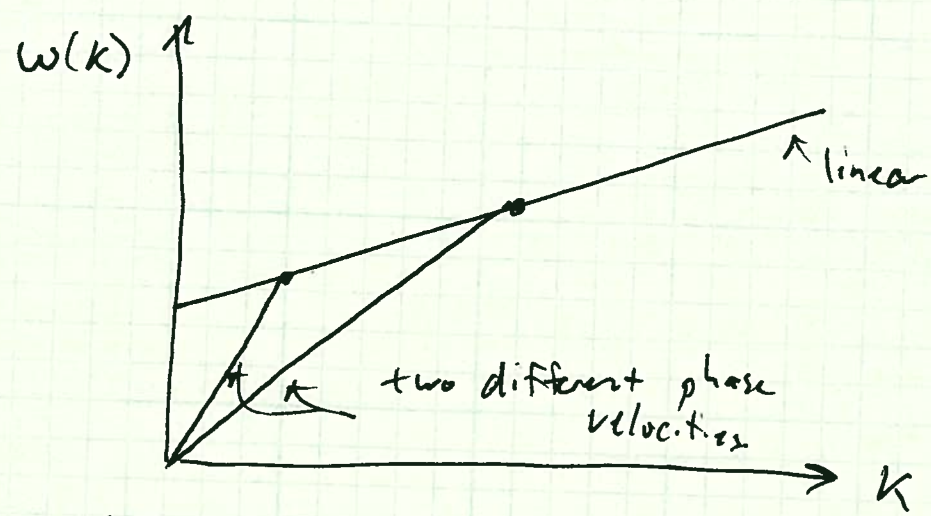
A perfect perfect travelling wave cannot be used to communicate. Because it is a perfect wave, it extends in time to (+) and (−) infinity, and ~~to~~ in space to (+) and (−) infinity. To ~~the~~ communicate a message, I would need to alter the wave in some way: turn it off, make it larger, change its frequency, etc. But doing any of these things would mean that the wave is no longer perfect, because it ~~the~~ would then have multiple frequency components. So to send a message, I will need multiple frequencies at my disposal.

But if the medium is dispersive, then the various frequency components will all travel at different velocities, and my message will disperse. So there will be some limit to how far I can communicate.

However, there is a clever way to send information a much longer distance by using a small range of frequencies to create a pulse.

As long as the dispersion relation is linear over that range of frequencies, we can make a pulse which travels forever.

To illustrate, imagine that our dispersion relation is linear, but not directly proportional:

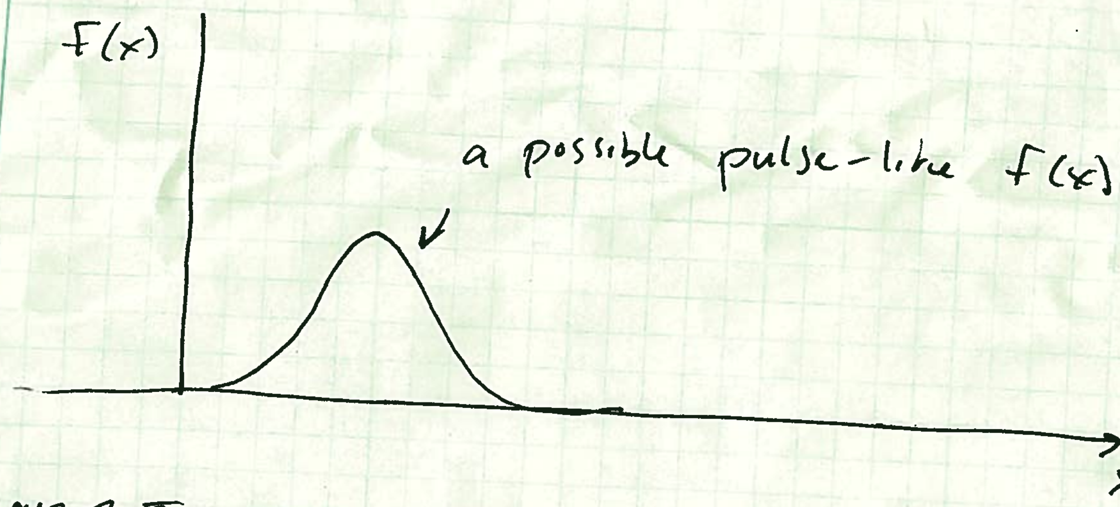


Since different waves have different phase velocities, this system is dispersive.

Now I create a pulse-like "envelope function" composed of a range of wave numbers.

$$f(x) = \text{a pulse-like function} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{F(k)}_{\substack{\uparrow \\ \text{the} \\ \text{Fourier transform}}} e^{ikx}\, dk$$

f(x) could be a gaussian pulse, for example:



~~Now I~~ ~~multiply~~

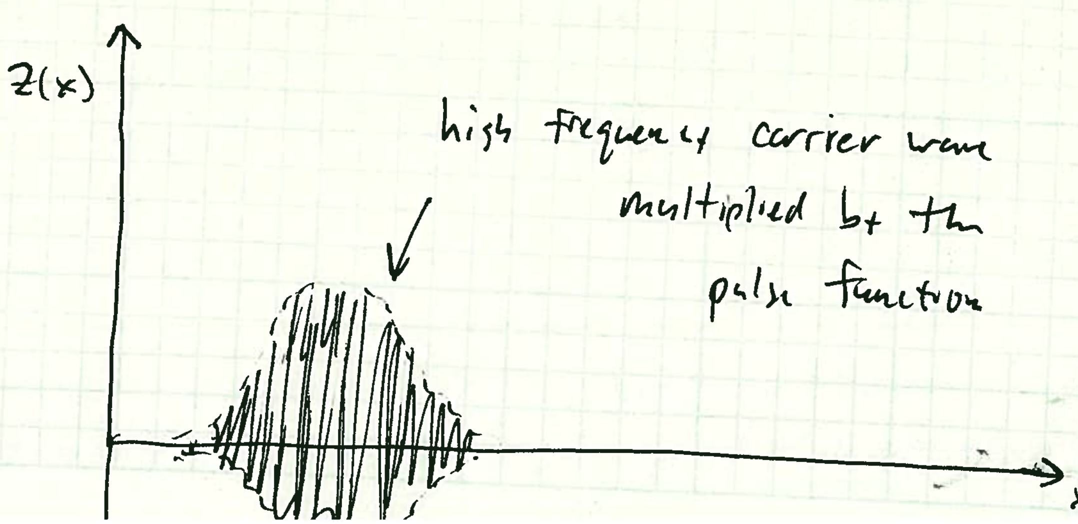
My claim is that I can make this pulse <u>propagate in time forever</u> by multiplying f(x) by a high frequency perfect travelling wave. The high frequency wave is known as the "carrier wave"

So let

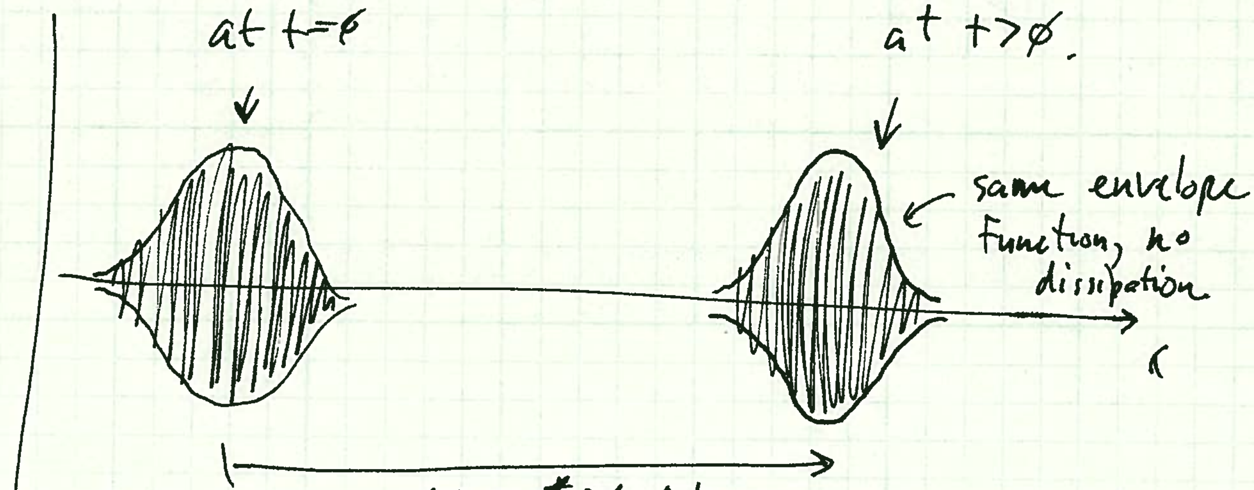
$$Z(x) = (\text{pulse}) \times (\text{carrier}) = f(x)\, e^{ik_c x}$$

where k_c = wave number of the high frequency carrier wave

Now $Z(x)$ looks like



Claim: The Pulse propagates with an ~~exact~~ envelope function which does not dissipate:



$\Delta x = v_g \Delta t$

"group velocity"
"the speed at which the envelope propagates".

If this claim is true, then the pulse propagation will be described mathematically as

$$Z(x) = f(x)e^{ik_c x} \text{ at } t=0$$

we would like to prove this → $$\boxed{Z(x,t) = f(x-v_g t)\, e^{i(k_c x - \omega_c t)}} \text{ at } t>0.$$

v_g: "group velocity"

$\omega_c = k_c v_{phase}$

$f(x-v_g t)$ describes the envelope moving at the group velocity without changing its shape

Now we prove this:

Substitute the Fourier expression for the pulse.

$$z(x) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k) e^{ikx}\, dk\right] e^{ik_c x}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k)\, e^{i(k+k_c)x}$$

Trick 1: Let $k' = k + k_c$.

Then $k = k' - k_c$, and we have

$$z(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk'\, F(k' - k_c)\, e^{ik'x}$$

integrate over k' now.

~~This equation says that the Fourier Transform of $z(x)$ is $F(k'-k_c)$.~~

But k' is just a variable of integration. We can re-name it k if we wish:

$$z(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ikx}$$ ← The Fourier Transform expression for $z(x)$.

Now we see that the Fourier Transform of $z(x)$ is $F(k-k_c)$.

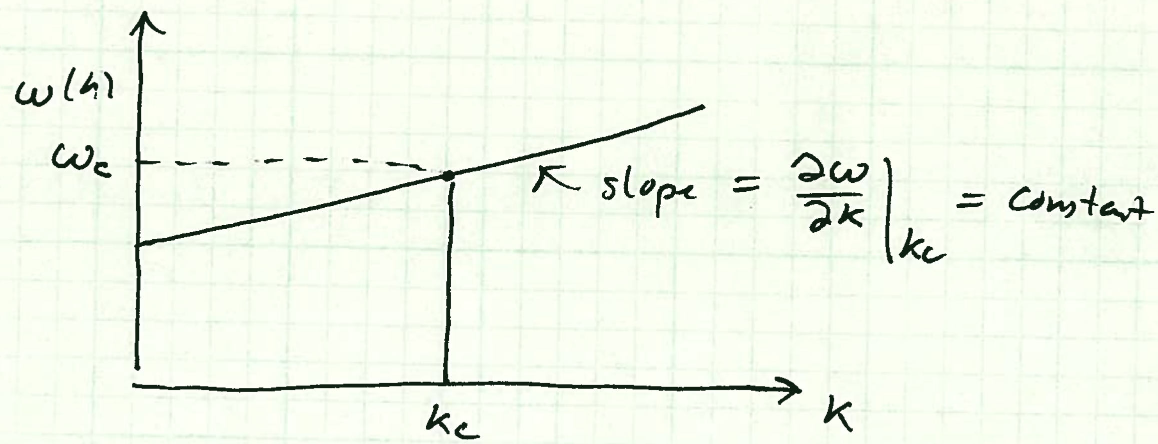
Let's make $z(x)$ move forward in time. To do that we multiply each travelling wave component by $e^{-i\omega(k)t}$:

$$z(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ikx}\, e^{-i\omega(k)t}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ik\left(x - \frac{\omega(k)}{k}t\right)}$$

Now we use our basic assumption: $\omega(k)$ is linear in k:

$$\omega(k) = \omega_c + (k-k_c)\left.\frac{\partial\omega}{\partial k}\right|_{k_c}$$



We call $\left.\frac{\partial\omega}{\partial k}\right|_{k_c} = V_g =$ "group velocity".

so $\omega(k) = \omega_c + (k-k_c)V_g$.

Then our travelling wave is

$$z(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ik\left(x-\left(\frac{\omega_c}{k}+\frac{kV_g}{k}-\frac{k_cV_g}{k}\right)t\right)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, F(k-k_c)\, e^{ikx}\, e^{-i\omega_c t}\, e^{-ikV_g t}\, e^{ik_cV_g t}$$

<u>Trick 2:</u> Let $k'' \equiv k-k_c$. Then $k = k''+k_c$. Therefore

$$z(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk''\, F(k'')\, e^{i(k''+k_c)x}\, e^{-i\omega_c t}\, e^{-i(k''+k_c)V_g t} \cdot e^{ik_cV_g t}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk''\, F(k'')\, \underbrace{e^{i(k_cx-\omega_c t)}}_{\text{does not depend on } k''}\, e^{ik''(x-V_g t)}$$

$$z(x,t) = e^{i(k_c x - \omega_c t)} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk'' F(k'') e^{ik''(x - v_g t)} \right]}$$

This is the Fourier Transfer of the envelope function $F(x - v_g t)$

$$\boxed{z(x,t) = F(x - v_g t)\, e^{i(k_c x - \omega_c t)}}$$

This is what we set out to prove: the envelope function $F(x)$ propagates without changing its shape: $F(x) \rightarrow F(x - v_g t)$. The speed of envelope progagation, known as the group velocity, has been determined to be

$$\boxed{v_g = \text{"group velocity"} = \frac{\partial \omega}{\partial k}(k = k_c)}$$

This result is important because <u>any</u> dispersion relation will be <u>approximately</u> linear over a small range of k. So pulses can be sent through any dispersion medium, as long as we use a sufficiently small range of k to make our pulses.

Example: Quantum free particle