Energy loss $Q \equiv \frac{\omega_0}{\gamma}$ = unitless & very large for lightly damped oscillators.

For lightly damped oscillators

$$Q = \frac{1}{\text{Fraction of energy lost in time } t = \frac{1}{\omega_0}} = \frac{2\pi}{\text{Fraction of energy loss in one period}}$$

Energy loss: $E(t) = E_0 e^{-\gamma t}$ = $KE(t) + U(t)$ for mechanical oscillators

$= U_E(t) + U_B(t)$ for electrical oscillators

AC circuits

Voltage rules:
$|V_C| = |\frac{1}{C} Q|$ Capacitor
$|V_L| = |L \frac{dI}{dt}|$ Inductor
$|V_R| = |IR|$ Resistor

Simple LC circuit: $\omega_0 = \frac{1}{\sqrt{LC}}$ (simple harmonic oscillator)

Impedances:
$Z_R = R$
$Z_L = i\omega L$
$Z_C = \frac{-i}{\omega C}$

Series Combination: $Z_{series} = Z_1 + Z_2$
Parallel Combination: $Z_{parallel} = \left[(Z_1)^{-1} + (Z_2)^{-1}\right]^{-1}$

For any element or combination of elements,

$$\vec{V} = \vec{I} Z \leftarrow \text{impedance, possibly complex.}$$

Voltage phasor — current phase

~~If Z~~ In general Z is complex, which means that there is a phase offset between $\vec{V}$ & $\vec{I}$.
The ~~ratio~~ ratio of $\frac{V_0}{I_0} = |Z|$.

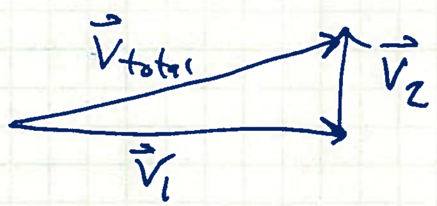
$$\vec{V}_L = \vec{I}_L \underbrace{(i\omega L)}$$

← this means V_L <u>leads</u> I_L by 90°.

$$\vec{V}_C = \vec{I}_C \underbrace{\left(\frac{-i}{\omega C}\right)}$$

← this means V_C <u>lags</u> I_C by 90°

<u>Phasor Diagrams</u> show the geometric relationships between $\vec{V}$ & $\vec{I}$ for a circuit.
This can be very useful when combined with a sum rule like $\vec{V}_{total} = \vec{V}_1 + \vec{V}_2$ or $\vec{I}_{total} = \vec{I}_1 + \vec{I}_2$



$\therefore |V_{total}| = \sqrt{V_1^2 + V_2^2}$

Normal Mode: In a multiparticle system, a normal mode is a type of motion where all particles oscillate at the same frequency.

- The number of normal modes is equal to the number of particles.
- Each normal mode goes at its own frequency.
- The general solution is a sum over normal modes:

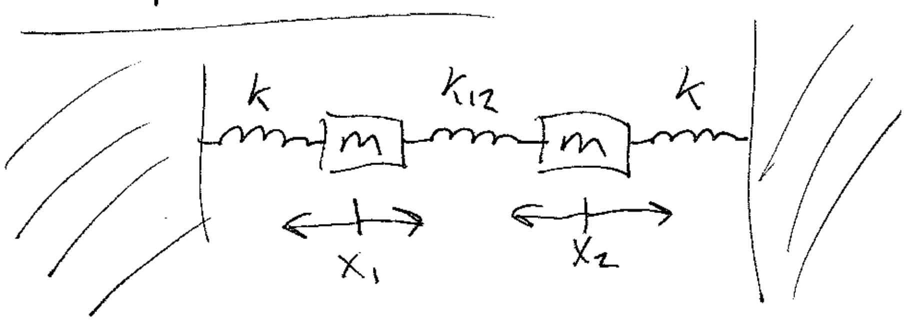
$$\vec{x}(t) = \sum_{n=1}^{2} c_n \vec{q}_n e^{i\omega_n t} \quad , \text{ for a 2 particle system.}$$

The expansion coefficients $\{c_n\}$ are determined by the initial conditions and can be calculated using "Fourier's Trick":

$$c_n \equiv a_n + i b_n, \quad a_n = \frac{\vec{x}_0 \cdot \vec{q}_n}{|\vec{q}_n|^2} \quad , \quad b_n = -\frac{\vec{v}_0 \cdot \vec{q}_n}{\omega_n |\vec{q}_n|^2}$$

where $\vec{q}_n$ are the normal mode eigenvectors and $\vec{x}_0$ and $\vec{v}_0$ are the initial position and velocity.

2 coupled oscillators



Two normal modes:

$\vec{q}_1 = (1,1)$ = "symmetric mode"

$\vec{q}_2 = (1,-1)$ = "anti-symmetric mode"

Eqs. of Motion:

$$m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 = \emptyset$$

$$m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = \emptyset$$

Solution:

$$\left(x_1(t),\, x_2(t)\right) = c_1(1,1)e^{i\omega_1 t} + c_2(1,-1)e^{i\omega_2 t}$$

For this system we called

$$\omega_1 = \omega_{small} = \sqrt{k/m}$$

$$\omega_2 = \omega_{Large} = \sqrt{\frac{k+2k_{12}}{m}}$$

Normal Mode: In a multi-particle system, a normal mode is a type of motion where all particles oscillate at the same frequency.

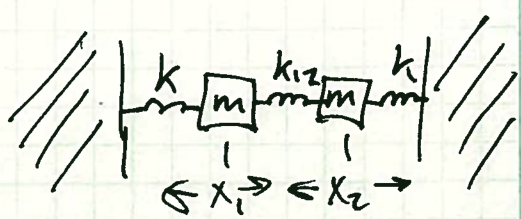
- The number of normal modes is equal to the number of particles.
- Each normal mode goes at its own frequency.
- The general solution is a sum over normal modes:

$$\vec{x}(t) = \sum_{n=1}^{N} a_n \vec{q}_n e^{i\omega_n t}$$

where $\vec{q}_n = (q_{1n}, q_{2n}, \ldots\ldots, q_{Nn}) = n^{th}$ normal mode

= "normal mode eigenvector"

2 coupled oscillators $(N=2)$



Two normal modes:

$\vec{q}_1 = (1, 1)$ = "symmetric mode"

$\vec{q}_2 = (1, -1)$ = "anti-symmetric mode"

Then $\vec{x}(t) = a_1 (1,1) e^{i\omega_1 t} + a_2 (1,-1) e^{i\omega_2 t}$

$$\downarrow$$

$$(x_1(t), x_2(t)) = a_1 (1,1) e^{i\omega_1 t} + a_2 (1,-1) e^{i\omega_2 t}$$

Eqs. of Motion

$m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 = 0$

$m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = 0$

For this system we called $\omega_1 = \omega_{small} = \omega_s$
and $\omega_2 = \omega_{Large} = \omega_L$

The frequencies are $\omega_1 = \omega_s = \sqrt{\dfrac{k}{m}}$

$$\omega_2 = \omega_L = \sqrt{\frac{k + 2k_{12}}{m}}$$

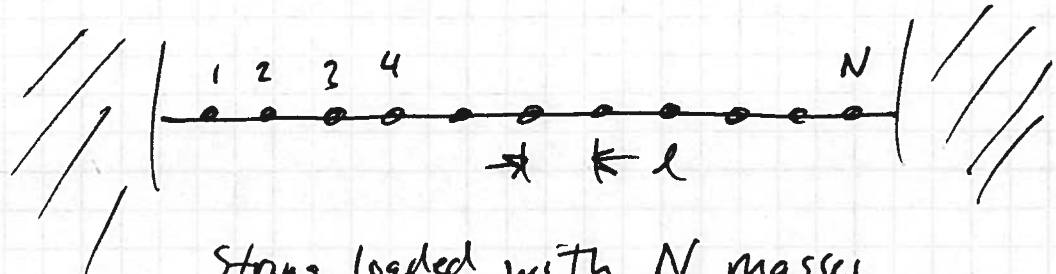
Explicitly the solution is

$$x_1(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$
$$x_2(t) = a_1 e^{i\omega_1 t} - a_2 e^{i\omega_2 t}$$

or, taking the real part,

$$x_1(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$$
$$x_2(t) = a_1 \cos(\omega_1 t) - a_2 \cos(\omega_2 t)$$

N - Coupled oscillators – The loaded string.



String loaded with N masses, each of mass (m).

String tension $= T$.

Transverse oscillations: Each mass has y-displacement (y_p).

String is fixed at each end, so $\left.\begin{array}{l} y_{p=0} = 0 \text{ and} \\ y_{p=N+1} = 0 . \end{array}\right\}$ boundary conditions

Equation of motion for mass (p):

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

where $\omega_0^2 = T/ml$.

Normal mode solutions:

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin\frac{Nn\pi}{N+1} \right)$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

We also wrote the $\vec{q}_n$ vectors on a particle-by-particle basis as

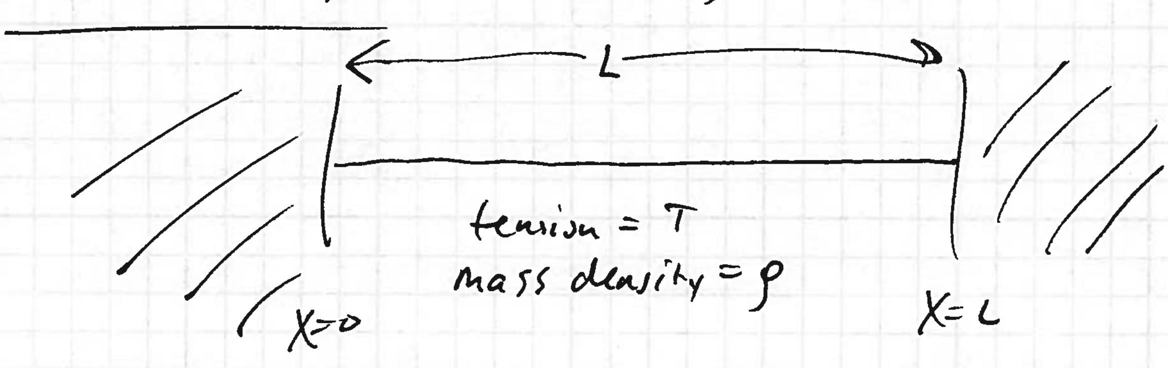
$$A_{pn} = \cancel{A} \sin\left(\frac{pn\pi}{N+1}\right)$$

which mass

which normal mode

The solution is the same for longitudinal oscillations, except the motion is in the x direction, rather than the y direction

Continuous Systems - string fixed at $x=0$ & $x=L$



Equation of Motion:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}}$$ Classical Wave Equation.

Normal Mode Solutions

$$y_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\omega_n = \sqrt{\frac{T}{\rho}}\,\frac{n\pi}{L} \quad , \; n = 1, 2, 3, \ldots \infty .$$

General Solution:

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

Fourier's Trick

Once the normal modes and normal frequencies of a system are known, the only thing that remains is to find the expansion coefficients to describe the system at $t=0$. We use Fourier's Trick to do this.

For the loaded string, Fourier's Trick says

$$a_i = \frac{\vec{y}_0 \cdot \vec{q}_i}{|\vec{q}_i|^2}$$ where $\vec{y}_0$ is the set of initial positions $(y_1(t=0), y_2(t=0), \ldots\ldots)$

~~For~~ For the continuous string fixed at $x=\phi$ and $x=L$, Fourier's Trick says

$$c_n = \frac{2}{L}\int_0^L y(x,t=\phi)\,\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier's Trick depends upon the fact that the eigen vectors are orthogonal:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\,\delta_{nm}$$ for the continuous string

and $$\sum_{j=1}^{N}\sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \left(\frac{N+1}{2}\right)\delta_{nm}$$ for the discrete loaded string.

Mathematics of Fourier Series & Fourier Transforms

Any periodic function with period $2L$ can be written

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

This same thing can be written in complex notation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx$ ← Fourier's Trick

The two forms can be converted into each other:

$$a_n = c_n + c_{(-n)}$$
$$b_n = i(c_n - c_{(-n)})$$
$$a_0 = c_0 + c_0 = 2c_0$$

and
$$c_n = \begin{cases} \frac{1}{2}(a_{(-n)} + i b_{(-n)}) & , \text{ for } n < 0 \\ \frac{1}{2}a_0 & , \text{ for } n = 0 \\ \frac{1}{2}(a_n - i b_n) & , \text{ for } n > 0 \end{cases}$$

The complex form is more compact and elegant. It also generalizes to the case where $f(x)$ is <u>no longer periodic</u> (period $L \to \infty$):

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx} \quad \text{(non-periodic } f(x)\text{)}$$

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$ ← "Plancherel's Theorem"

But it's really just another example of Fourier's Trick.

Travelling waves – Continuous String with no boundaries

If we have a continuous string with no boundary conditions (no walls), then we can have travelling wave solutions

$$y(x,t) = A\sin(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda}, \quad \lambda = \text{wavelength}.$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The peaks and troughs move forward at the "phase velocity"

$$V_{phase} = v = \frac{\omega}{k} = \lambda f$$

The equation of motion is still the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

And the travelling waves satisfy this as long as

$$v = \sqrt{\frac{T}{\rho}} = \text{phase velocity}.$$

So we could write the Eq. of Motion as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

The general solution for a continuous string with no boundaries is a sum over all travelling waves. But since any wavelength is allowed, we have to sum over a continuum of k-values:

$$y(x, t=0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$$

where $$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, F(x)\, e^{-ikx}$$

Then, as time goes forward, each normal mode (e^{ikx}) gets its own phase factor $(e^{i\omega t})$

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{i(kx+\omega t)}$$