Homework #7 - Phys 273

Questions highlighted in yellow are optional.

1) Consider a lopsided triangle function defined from x = 0 to x = 1 meter:

$$f(x) = \begin{cases} x & 0 \le x \le d \\ \dfrac{d}{1-d}(1-x) & d \le x \le 1 \end{cases}$$

In this definition, (d) is some unitless fraction between zero and one.

a) Sketch this function (or draw it with a computer) for the case where d = 0.75.

b) This function can be represented by a Fourier Sine Series:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

where, again, L = 1 meter. According to Fourier's trick, the coefficients can be calculated:

$$a_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

Calculate these coefficients. Hint: the correct answer comes out to be:

$$a_n = \frac{2}{(1-d)(n\pi)^2}\sin(n\pi d)$$

c) Let's stick with d = 0.75 for the remainder of this problem. Use a computer to draw the Fourier Series, but only keeping the first term in the sum.

d) Now draw the series keeping the first two terms, and the first three terms. (This amounts to two additional plots.) (Optional: keep the first 100 terms in the sum.)

e) Let's consider this lopsided triangle to be the initial state at t = 0 of a continuous string. Let the tension in the string be 10 N, and the mass density be 0.1 kg/meter. Draw the shape of the string at t = 0.005 seconds, 0.010 seconds, and 0.015 seconds, keeping the first three terms in the sum. (Optional: keep the first 100 terms in the sum.)

2) If a function f(x) is periodic, with period 2L, and if it is square integrable between (-L,L), then we can represent it as a linear combination of sine and cosine functions (a Fourier Series):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Suppose we want to represent a square wave function as a Fourier Series:

$$f(x) = \begin{cases} -1, -L < x < 0 \\ 1, 0 < x < L \end{cases}$$, periodic with period 2L.

a) Sketch this function.
b) Calculate (a_0) for this square wave.

c) Calculate the $\{a_n\}$, for this square wave.

d) The answer to part (c) is very simple. Why?

e) Calculate the $\{b_n\}$ for this square wave.

f) Use a plotting program to graph the Fourier Series on the interval (-3L, 3L) keeping the first three terms in the sum. (Optional: keep the first 100 terms in the sum.)

3) Consider this parabolic function:

$$f(x) = x(L - x), \quad 0 \le x \le L$$

a) Write this function as a Fourier sine series by calculating the correct set of expansion coefficients. Consider only the interval of x = 0 to x = L.

b) Make a plot of the Fourier sine series from x = 0 to x = L, keeping three non-zero terms in the sum. Also put the exact parabolic function on the same plot.

4) The classical wave equation and its general solutions are given in the lecture notes. Show that the general solution is correct by explicitly substituting it into the equation of motion.

5) Consider the set of functions $\{e^{in\pi x/L}\}$, where (n) is any positive or negative integer. Show that these functions are orthogonal to each other over the interval (-L, L) by evaluating this integral:

$$\int_{-L}^{L} \left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right)dx.$$

Evaluate the integral for the case where n = m and the case where n ≠ m.