Physics 273 - Homework #6

1) For the loaded string, the amplitude relationships which define the normal modes are described by the expression:

$$A_{pn} = \sin\left(\frac{pn\pi}{N+1}\right)$$

where (p) tells us which mass we are talking about, (n) tells us which normal mode we are talking about, and N is the total number of masses on the string. It is convenient to re-write this expression in a vector notation:

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin\left(\frac{Nn\pi}{N+1}\right)\right)$$

In this expression, the first component of the vector describes mass #1, the second component describes mass #2, ect. Since there are N normal modes, there will be N such vectors. These are the normal mode eigenvectors.

a) Write down in explicit numerical form the four vectors for the N = 4 case. Please give a numerical value for each component of the vectors, accurate to three decimal places.

b) Calculate these three dot products:
$\vec{q}_1 \cdot \vec{q}_2$, $\vec{q}_1 \cdot \vec{q}_3$, $\vec{q}_1 \cdot \vec{q}_4$

2) Consider a loaded string consisting of three particles of mass (m) regularly spaced on the string. At t = 0 the center particle is displaced a distance (a) from its equilibrium position. (The other two particles are located at their equilibrium positions.) We release all three particles with an initial velocity of zero.

a) Apply these initial conditions to the solution of the loaded string (which we found in class) to find the position of all three particles as a function of time.

b) Let the string tension be T = 10 N, m = 1 kg, and let the distance between the masses on the string be 0.1 m. Also let the initial displacement of mass 2 be 0.01 m. Make a plot of the positions of all three masses from t = 0 to t = 10 seconds. Please put $x_1(t)$, $x_2(t)$, and $x_3(t)$ on the same plot.

3) Let's consider a continuous string with the triangular initial shape. The string is mounted between two fixed walls at x = 0 and x = L, and its equation of motion is the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

and the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}, \text{ where } \omega_n = \sqrt{\frac{T}{\rho}}\frac{n\pi}{L},$$

and the $\{C_n\}$ are some set of coefficients which are determined by the initial conditions. In class we will calculate the real and imaginary parts of the $\{C_n\}$ for the case where the string has a triangular shape with height = (h) at t = 0:

$$y(x,t=0) = \begin{cases} \dfrac{2hx}{L} & 0 \le x \le L/2 \\ \dfrac{2h(L-x)}{L} & L/2 \le x \le L \end{cases}$$

and zero initial velocity:

$$\dot{y}(x,t=0) = 0.$$

The result is

$$\mathrm{Re}(C_n) \equiv a_n = \frac{8h}{n^2\pi^2}(-1)^{(n-1)/2} \text{ for odd (n) and } a_n = 0 \text{ for even (n), and}$$

$$\mathrm{Im}(C_n) \equiv b_n = 0 \text{ for all (n).}$$

Please turn in a plot for the following:

a) First consider the solution at t = 0. Let the initial height of the triangle be h = 0.5 meters, and the length of the string be 10 meters. Use a computer to draw the solution, but only including <u>the first three non-zero terms</u> of the infinite sum. (Optional: if it's not too much trouble, keep the first 100 non-zero terms).

b) Now keep the first three (or 100) non-zero terms, as in part (a), but this time we will allow the solution to evolve in time. Let the tension in the string be 10 N and the mass density be 0.1 kg/meter. Draw the shape of the rope at t = 0.1, t = 0.2, and 0.3 seconds, keeping just the first three (or 100) terms in the sum.

4) **Ortho-normality of Sine functions.** The Kronecker Delta (δ_{nm}) is defined to be equal to 0 for n $\neq$ m, and equal to 1 for n = m. Given this definition, show that

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \delta_{nm}.$$

where (n) and (m) are integers. Explicitly evaluating the integral for

a) the n = m case.

b) the n $\neq$ m case.

Hint: You may use this trigonometric identity: $\sin(u)\sin(v) = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$.