Physics 273 - Homework #5

Questions highlighted in yellow are optional.

1) Consider the two coupled mechanical oscillators shown in Figure 1. Each of the masses are connected to fixed walls with a spring (k). The masses are connected to each other with a different spring (k_{12}).

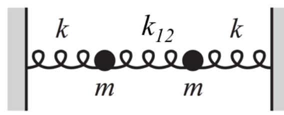


Figure 1

a) Suppose we hold one of the two masses fixed, so it cannot move. What is the natural frequency of the other mass?

b) Now we release the second oscillator. Compare the natural frequency from part (a) to the two normal mode frequencies of the double oscillator. Are they smaller or larger? Explain.

c) What happens to the two normal mode frequencies of the system in the limit where the coupling spring constant (k_{12}) becomes very large and very small? Explain.

2) Consider again the two coupled mechanical oscillators shown in Figure 1 above. Suppose that our initial conditions are that the first oscillator has an initial velocity of (v_0) and an initial position of zero, and the second oscillator starts with an initial velocity and position of zero.

a) Apply these initial conditions to find $x_1(t)$ and $x_2(t)$.

b) Let k = 1 N/m, k_{12} = 0.25 N/m, m = 1 kg, and v_0 = 1 m/s. Make a plot of the positions of both masses from t = 0 to t = 30 seconds. Please put both $x_1(t)$ and $x_2(t)$ on the same plot.

3) **Ringing and damping of a mechanical oscillator (numerical).** Make a copy of your numerical solution to Homework #4 problem #2 (forced oscillator with damping). In this problem we will change the forcing function to the following step function (or square wave function):

$$F(t) = \begin{cases} 1.0 \text{ Newtons}, & 0 < t < 10 \text{ s} \\ 0.0 \text{ Newtons}, & 10 \text{ s} < t < 20 \text{ s} \end{cases}$$

This forcing function repeats itself thereafter with a period of 20 seconds. Hint: If you are using excel, one way to implement this forcing function in your numerical calculation is to create a new column which contains the value of the forcing function at each moment in time. You can then reference this column when calculating the acceleration.

a) Let x0 = 0.0 m, v0 = 0.0 m/s, m = 1 kg, k = 30 N/m, and b = 2 N/(m/s). Use a time step of 0.01 seconds, and calculate for 4000 steps (a total of 40 seconds). Print out a plot the position of the oscillator as a function of time.

b) In your solution you should see a phenomena called "ringing". Measure the angular frequency of the ringing, and compare it to the oscillation frequency that you would expect for this oscillator.

c) The frequency of a damped oscillator is given by

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4}$$

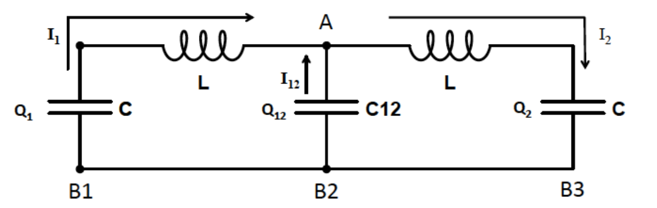
Suppose we increase the drag coefficient (b or γ) until $\omega_d = 0.0$. This condition is called "critical damping", and it is the condition for all oscillations to be eliminated. Calculate the value of the drag coefficient (b) which achieves critical damping for the oscillator parameters described in part (a).

d) Starting with a drag coefficient of b = 2 N/(m/s), increase (b) one unit at a time up to 20 N/(m/s), and observe the effect on the oscillator's position as a function of time. Now set (b) equal to the critical value that you calculated in part (c), and print out a plot of the oscillator's position for this situation.

Comment: In real mechanical and electrical systems where oscillations are undesirable, the components will often be chosen so that the system is critically damped. For example, a bridge might be designed to be critically damped to prevent large oscillations in the event of an earthquake.

e) When the drag coefficient is increased beyond the critical value, we say that the system is "over-damped". Just like critically damped systems, over-damped systems also do not oscillate, however, they take longer to return to equilibrium after a shock. To see the response of an over-damped system, set b = 50 N/(m/s), and print out a plot of the oscillator's position as a function of time.

5) Consider two coupled LC oscillators:



This circuit is the electrical analog of the coupled mechanical oscillators that we studied earlier in the semester (see Homework #5).

a) To analyze this circuit, start by rewriting the known equations of motion for the coupled mechanical oscillators in the following form:

$$\ddot{x}_1 + \omega_0^2(1+\alpha)x_1 - \omega_0^2\alpha x_2 = 0$$
$$\ddot{x}_2 + \omega_0^2(1+\alpha)x_2 - \omega_0^2\alpha x_1 = 0$$

Identify the constants (ω_0) and (α) in terms of (k), (k_{12}), and (m), and write down the two normal mode frequencies for the mechanical oscillator in terms of (ω_0) and (α). (You don't need to solve the equation of motion to do this; just write the known expressions for (ω_1) and (ω_2) in terms of (ω_0) and (α)).

b) Consider the three currents (I_1, I_2, and I_{12}) which meet at junction A in the above circuit diagram. Use Kirchoff's current sum rule to write down the relationship between these currents. Note the sign convention indicated in the diagram: the direction of each arrowhead gives the direction of positive current.

c) Use the relationship between the currents and the charges in the circuit to convert the current sum rule from part (b) into a sum rule for the charges (Q_1, Q_2, and Q_{12}).

d) Consider the voltage drop from junction A to junctions B_1, B_2, and B_{12}. Since these three voltage drops must be the same, we have $V(AB_1) = V(AB_2)$ and $V(AB_3) = V(AB_2)$ Using the voltage drop rules for capacitors and inductors, write these two equations in terms of the currents, charges, L, C, and C_{12}.

e) Use the charge sum rule from part (c) to eliminate (Q_{12}) from the two voltage drop equations.

f) Take the time derivative of both equations to get the coupled equations of motion for (I_1) and (I_2). Show that these equations of motion can be written in the same form as the mechanical oscillator equations from part (a), and identify the constants (ω_0) and (α) for the coupled LC oscillators.