**Physics 273 - Homework #3**

1) The equation of motion for a simple harmonic oscillator including damping and a driving force is:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t)/m$$

We can solve this equation by pretending that (x) is a complex variable (x = z) and that F(t) is a complex function. This gives us a fictitious equation of motion:

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = F(t)/m \text{, F(t) = complex.}$$

a) We are allowed to solve the fictitious equation of motion because its real and imaginary parts both exactly replicate the true equation of motion. Show that this is true by letting (z = x + iy) and F(t) = $F_x$(t)+ i$F_y$(t). (When we solve the fictitious equation, we are solving the true equation twice, once on the real axis, once on the imaginary axis.)

b) This trick does not work for a non-linear equation of motion. For example, suppose the equation of motion is

$$\ddot{x} + \frac{\omega_0^2}{x_0} x^2 = F(t)/m,$$

$x_0$ = some constant with units of distance.

Repeat part (a) and show that we do not get the right equation of motion by pretending that (x) is complex.

2) **Numerical solution of a pendulum.** The equation of motion for a simple pendulum is

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

This equation cannot be solved analytically for θ(t). For small θ, however, we can approximate the equation as

$$\ddot{\theta} + \frac{g}{\ell}\theta \approx 0$$

since sin(θ) ≈ θ for small θ.

a) In the small angle approximation, what is the expected natural frequency of the pendulum if its length is 0.1 meters and g = 9.8 m/s$^2$? Assuming that the amplitude is always small, does the natural frequency depend on the amplitude?

b) Numerically we can say that

$$\ddot{\theta}_{i+1} = -\frac{g}{\ell}\sin\theta_i$$

This equation is true even for large amplitude oscillations, at least to within the numerical precision of the calculation. To solve this equation numerically for θ as a function of time, make a copy of your solution from problem #6 of Homework #2 (the numerical simple harmonic oscillator) and modify it. Use 0.1 meters as the length of the pendulum, and 9.8 m/s$^2$ as the acceleration due to gravity. As your initial conditions, assume that the pendulum is released from rest with a starting angle of $\theta(t=0) = 0.1$ radians. Use a time step of 0.002 seconds, and calculate for 1000 time steps (a total of 2 seconds). Print out a plot of $\theta$ as a function of time.

c) Compare the angular frequency that you observe in your numerical calculation to the expected value from part (a).

d) Try changing the starting angle to 0.2 radians, 0.3 radians, continuing in 0.1 radian steps up to 1.5 radians (a total of 16 cases). For each of these cases, record the angular frequency that you observe in your solution. Make a plot of angular frequency as a function of starting angle for these 16 cases.

3) **Numerical solution of a damped oscillator.** Make another copy of your numerical simple harmonic oscillator from Homework #2. This time we will modify it by adding a damping term. The equation of motion is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

where $\gamma = b/m$, and b is the drag coefficient.

a) Suppose that the oscillator starts at $x(t=0) = 1$ meter, and it is released from rest. Solve the equation of motion numerically for x(t). Remember that at each step the velocity and position are derived from the acceleration, and the acceleration is determined by the equation of motion. Use a drag coefficient of 0.3 N/(m/s), a mass of 1 kg, a spring constant of 1000 N/m, and a time step of 0.001 seconds. Calculate for 1000 time steps (a total of one second), and print out a plot of x(t). Is the amplitude of the oscillation decreasing approximately linearly in time during the first second of its motion?

b) What is the Q factor of this oscillator?

c) Change the drag coefficient to 10 N/(m/s) and print out a new plot of x(t). Does the amplitude decrease approximately linearly during the first second under these conditions?

## Questions highlighted in yellow are optional.

1c) (continued from above): The fictitious solution to the simple harmonic oscillator with no damping or driving force is

$$x(t) = Ae^{i(\omega_0 t+\delta)}$$

The total mechanical energy of the oscillator is

$$E = KE + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Show that if we use the fictitious solution in this expression we get the wrong answer for the total energy. Again, the problem here is that we are applying a non-linear operation (squaring), and our trick of using complex notation doesn't work quite right in this case.

1d) (continued from above): All is not lost, however. Show that if we re-interpret the expression for the total energy as follows, we get the correct answer after all:

$$E = KE + U = \frac{1}{2}m\left[\frac{\dot{x}\dot{x}^*}{2}\right] + \frac{1}{2}k\left[\frac{xx^*}{2}\right]$$

Here we have replaced the squaring operation with multiplication by the complex conjugate, and divided by two.

4) For this question, please construct two simple pendulua from string and tape, each of length = 1 meter. For pendulum #1, use a US quarter as the bob. For pendulum #2, use an empty aluminum soda can.

a) Calculate the expected natural frequency for the two pendula ($\omega_0$).

b) Measure the actual frequency for each of your two pendula. Please report your result in terms of angular frequency ($\omega$).

c) Your pendula are not ideal harmonic oscillators because they experience drag forces. Using a ruler to measure the amplitude of oscillation, determine the damping factor ($\gamma$) for each of your two pendula. (Here we are assuming that the drag force can be modeled by the simple expression $F_d = -bv$).

d) Look up the typical mass for a US quarter and for an empty aluminum can. Using the mass, convert your measurement of ($\gamma$) into a determination of the drag force constant (b). Please report your result in SI units.

e) Calculate the Q factor for each of your two pendula. Are they different? If so, why?