Phys 273 - Homework #1

1) Suppose that a particle of mass (m) is moving in one dimension under the influence of a force with potential energy

$$U(x) = V_0\left[-\frac{x}{b} + \left(\frac{x}{b}\right)^3\right]$$

where (V_0) is a positive constant with units of Joules and (b) is a positive constant with units of meters.

a) Sketch by hand or plot with a computer the potential energy as a function of (x). Note that (x) can be positive or negative.

b) Calculate the force as a function of (x).

c) Find the equilibrium position or positions in terms of (V_0) and (b). How many equilibrium positions are there?

d) Are the equilibrium positions you found in part (c) stable or unstable?

e) You should have at least one stable equilibrium position. Let this position be called (x0). Expand the potential function as a Taylor series about (x0), keeping terms up to second order in distance from the equilibrium position.

f) What is the frequency of small oscillations about (x0)? Please report the angular frequency (ω).

2) Consider a mass (m) hanging from a vertical spring attached to the ceiling. Let z = 0 be the vertical position where the spring exerts no force, and let positive z point upwards. Then the mass experiences a spring force of (-kz) and a gravitational force of (-mg).

a) Write down (or calculate) the potential energy function U(z) in terms of (m), (g), (k), and (z).

b) Let z0 be the equilibrium position where the mass experiences no net force. Calculate z0 in terms of (m), (g), and (k).

c) Let (a) stand for the displacement from the equilibrium position, so that z = z0 + a. Rewrite the potential energy function in terms of (m), (g), (k), and (a).

d) You now have the potential energy function written as an expansion about the equilibrium position. Determine the frequency of small oscillations about the equilibrium position. Please report the angular frequency (ω).

e) Is the frequency the same or different than that of a horizontal mass on a spring?

3) **Numerical solution of a ball in a uniform gravitational field.**

Many oscillating systems, including the simple harmonic oscillator, are described by equations of motion which can be can be solved exactly. In more advanced physics courses, however, and particularly in physics research, it is unusual to find a system for which an exact analytic solution is available. For these systems we must study their behavior by numerically solving the equations of motion.

In this problem you will numerically solve the equation of motion for a ball in a uniform gravitational field. In a numerical solution we step along in time in small increments, calculating at each step the acceleration, the velocity, and the position. If the acceleration, velocity, and position at time step number (i) are called a_i, v_i, and x_i, then the velocity and position at the next time step (i+1) will be:

$$v_{i+1} = v_i + a_{i+1} \cdot \Delta t$$
$$x_{i+1} = x_i + v_{i+1} \cdot \Delta t$$

In other words, the new velocity will simply be the old velocity plus the acceleration times the time step (Δt), and similarly for the new position. These equations follow simply from the definition of the velocity and acceleration.

Two more ingredients are needed for a complete numerical solution. First, we have to have a rule which tells us the acceleration at each moment in time. This is given by Newton's 2nd Law:

$$a_i = F_i / m$$

Since the force depends only on known variables, such as position, time, and velocity, it should be calculable at each moment in time. Secondly, to start the numerical solution we must specify the position and velocity at the first time step: x_0 and v_0. These are our initial conditions.

Finally, we also have to choose an appropriate time step (Δt). The time step should be chosen to be small enough that none of the variables change very much from one step to the next, or otherwise the numerical equations will be inaccurate. Roughly speaking, you should choose the time step so that the resulting plots of (x) vs. time and (v) vs. time appear reasonably smooth.

You may use any software or programming language that you like to solve this problem. Even a spreadsheet program such as Excel would be fine. I'll make just one comment regarding how to write your program: make sure that the constant parameters, such as the value the acceleration due to gravity (g), the mass (m), and the initial conditions (x_0 and v_0), are recorded in one and only one place in your program. This will make it easy to change these variables and see the effect on the results.

a) A ball is thrown upward from a height of 1 meter above the ground with an initial velocity of 3 meters per second. From your knowledge of classical mechanics and kinematics, calculate by hand the exact value of the highest point in its trajectory.

Hint: you can calculate this quickly using energy conservation. (Neglect drag due to air resistance, and assume that the acceleration due to gravity is 9.8 m/s^2. Also, let the height of the ground be defined as zero.)

b) Calculate exactly by hand how long will it take for the ball to strike the ground.

c) This problem is particularly easy to solve numerically, because the force is constant, and therefore the acceleration is constant. Use a time step (Δt) of 0.01 seconds, and calculate the velocity and position for at least 120 steps (a total of 1.2 seconds). Print out a plot of the height vs. time (or sketch it by hand).

d) Look through your time-stepping data, and find the maximum height of the ball, and the time that it takes to reach the ground. How do these values compare to what you calculated in parts (a) and (b)?

e) Suppose the initial upward velocity was only 1 m/s. Using your numerical calculation, how long would it take for the ball to strike the ground?

f) Suppose we want the ball to strike the ground at $t = 0.2$ seconds. Try different initial velocities until you find one that has the ball striking the ground between 0.19 and 0.2 seconds. What value of the initial velocity achieves this? Also print out a plot of height vs. time for this situation.