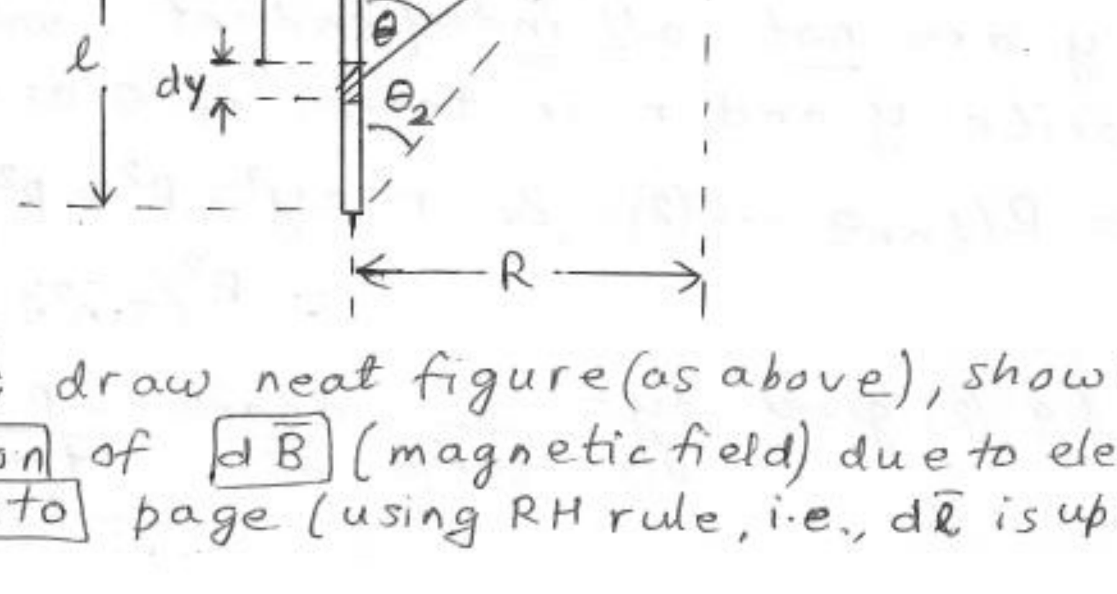
Magnetic field created by a straight, but finite length of current-carrying wire

(see also Example 28-11 of Giancoli)

- Consider the configuration shown below with current I in wire of length ℓ, with field point, P (i.e., where we want magnetic field, $\vec{B}$) at perpendicular distance R from wire and with d being distance from top of wire to point (origin, O) where perpendicular from P meets the extrapolated current direction
- Let θ denote angle between current direction and position vector $\vec{r}$ to field point from infinitesimal element dy of wire (positive y direction is vertical, but downwards)



Step "0": draw neat figure (as above), showing direction of $d\vec{B}$ (magnetic field) due to element dy, i.e., into page (using RH rule, i.e., $d\vec{\ell}$ is upward -

in direction of I, with $\bar{r}$ as shown, so that $d\bar{l} \times \bar{r}$ is into page)

Step $\boxed{I}$: we have magnitude of dB given by Biot-Savart law :

$$|d\bar{B}| = \frac{\mu_0}{4\pi} I \frac{\overbrace{dy}^{>0 \text{ if } y \text{ goes from } d \text{ to } l+d \text{ as shown}} \sin\theta}{r^2} \quad \ldots (1) \quad \text{(here } dl = dy\text{)}$$

(again θ is angle between $d\bar{l}$ & $\bar{r}$)

Step $\boxed{II}$ Use symmetry : here we see that direction of $d\bar{B}$ is into page for all elements dy, i.e., irrespective of y (since $d\bar{l}$ is always upward vs. $\bar{r}$ points "to right") So, effectively we have a "scalar" addition (integral) of $d\bar{B}$'s to do here

Step $\boxed{III}$ Integrate $|d\bar{B}|$ from Eq. (1): note that the 3 variables entering $|d\bar{B}|$ above, i.e., r, θ and y are not all independent : we choose to write y and r in terms of θ as follows

$$y = R/\tan\theta \quad \ldots (2) \quad \& \quad r^2 = y^2 + R^2 = R^2\left(\frac{1}{\tan^2\theta} + 1\right) = R^2/\sin^2\theta \quad \ldots (3)$$

Now, Eq. (2) gives
$$\frac{dy}{d\theta} = -\frac{R}{\tan^2\theta} \frac{d\tan\theta}{d\theta} = -\frac{R}{\tan^2\theta} \frac{1}{\cos^2\theta} = -R/\sin^2\theta \quad \ldots (5)$$

Plugging Eqs. (3) & (5) in Eq. (1) and integrating from θ_1 (corresponding to $y = d$) to θ_2 (i.e., $y = d + \ell$) gives

$$\text{(net) } B \text{ (again into page)} = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \overbrace{\left(\frac{-R}{\sin^2\theta}\, d\theta\right)}^{dy} \times \frac{\sin\theta}{\underbrace{\left\{R^2/\sin^2\theta\right\}}_{r^2}}$$

$$= -\frac{\mu_0 I}{4\pi R} \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta$$

$$= \frac{\mu_0 I}{4\pi R} \left(\cos\theta\right)\Big|_{\theta_1}^{\theta_2} = \frac{\mu_0 I}{4\pi R} \left(\cos\theta_2 - \cos\theta_1\right) \quad \ldots (6)$$

Finally, we can relate $\theta_{1,2}$ to d, ℓ, R using $\tan\theta_1 = R/d$ and $\tan\theta_2 = R/(d+\ell)$

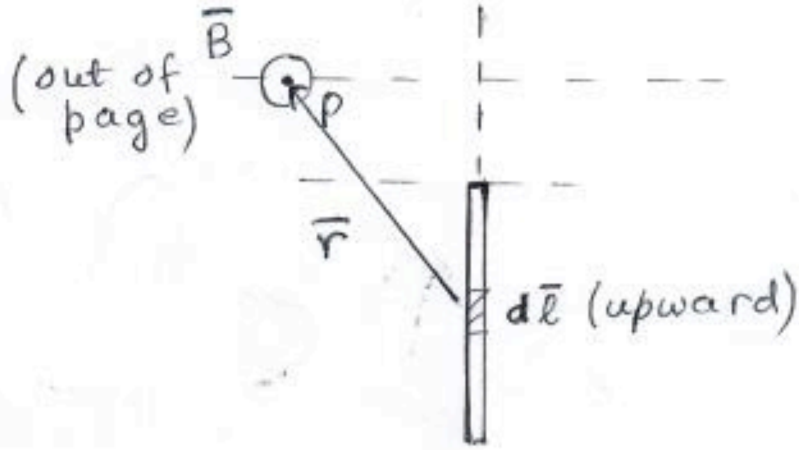
so that $\cos\theta_{1,2} = 1/\sqrt{(1 + \tan^2\theta_{1,2})} = \dfrac{1}{\sqrt{1 + \dfrac{R^2}{d^2 \text{ or } (d+\ell)^2}}}$

$$= \frac{d \text{ or } (d+\ell)}{\sqrt{R^2 + d^2 \text{ or } (d+\ell)^2}} \quad \ldots (7)$$

Plugging Eq. (7) into Eq. (6) gives

$$\boxed{\bar{B} \text{ at P (into page)} = \frac{\mu_0}{4\pi I R}\left(\frac{d+\ell}{\sqrt{R^2+(d+\ell)^2}} - \frac{d}{\sqrt{R^2+d^2}}\right)}$$

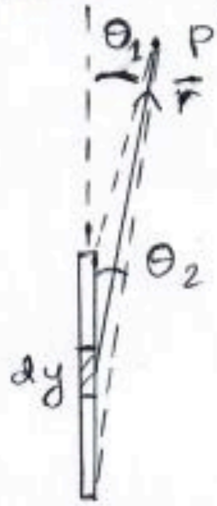
- One can also consider field point P being to the "left" of the wire: using $d\bar{l} \times \hat{r}$ it is easy to see that $\bar{B}$ at this point will be <u>out</u> of the page:



Similarly, considering other locations of P, we find that $\bar{B}$ is "circumferential", i.e., $\bar{B}$ lines are concentric circles (looking from top) centered on the wire.

step $\boxed{IV}$ Consider limiting cases, where answer is known from simpler arguments, e.g.

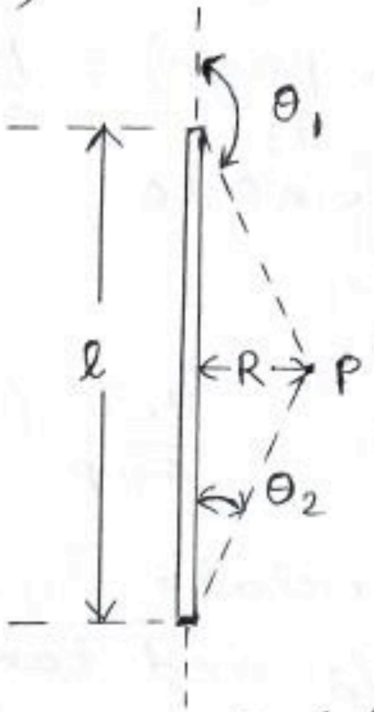
(a) $\boxed{\text{P is on the line containing the wire}}$, i.e., take limit $\theta_{1,2} \rightarrow 0$:



The general formula in Eq. 6 then gives $\boxed{\bar{B} = 0}$.

This was expected to "begin with": $d\bar{l}$ (always along the wire) and $\bar{r}$ (again from dy element to P) become parallel to each other in this limit (since $\bar{r}$ for all dy's aligns with wire)

(b) Wire becomes "infinitely" long : equivalently, $\boxed{R \ll l}$
P gets very close to wire, (but away from its ends) :



In this case, $\theta_1 \to \pi$ and $\theta_2 \to 0$: plugging into Eq. 6 gives

$$\boxed{B\begin{pmatrix}\text{circumferential,}\\ \text{distance R from}\\ \infty \text{ long wire}\end{pmatrix} = \frac{\mu_0 I}{2\pi R}}$$

as deduced (much) more simply from Ampere's law.