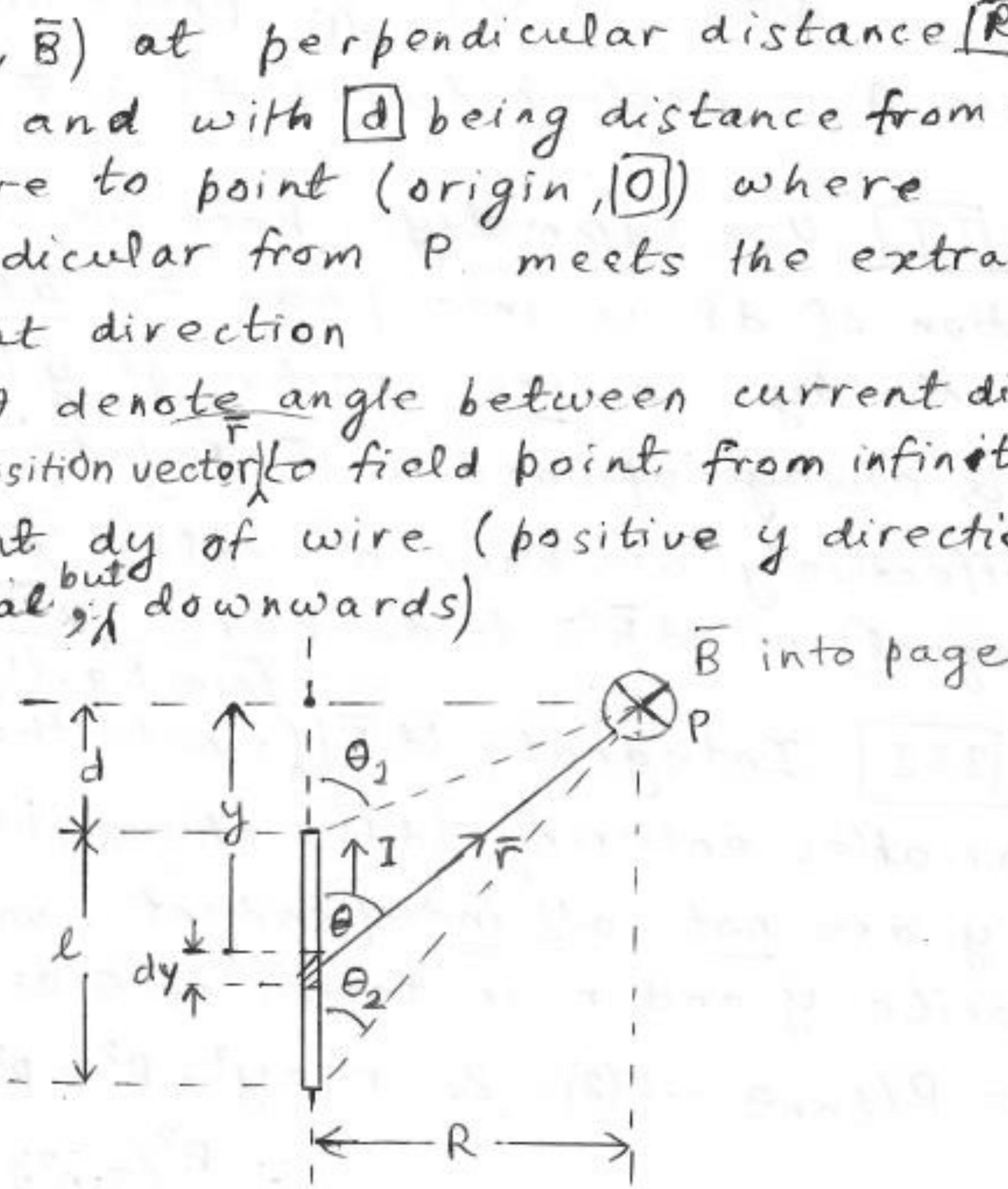
Magnetic field created by a straight, but finite length of current-carrying wire

(see also Example 28-11 of Giancoli)

- Consider the configuration shown below with current I in wire of length ℓ, with field point, P (i.e., where we want magnetic field, $\bar{B}$) at perpendicular distance R from wire and with d being distance from top of wire to point (origin, O) where perpendicular from P meets the extrapolated current direction
- Let θ denote angle between current direction and position vector $\bar{r}$ to field point from infinitesimal element dy of wire (positive y direction is vertical, but downwards)



Step "0": draw neat figure (as above), showing direction of $d\bar{B}$ (magnetic field) due to element dy, i.e., into page (using RH rule, i.e., $d\bar{\ell}$ is upward –

in direction of I, with $\bar{r}$ as shown, so that $d\bar{l} \times \bar{r}$ is into page)

Step $\boxed{I}$: we have magnitude of dB given by Biot-Savart law :

$$|d\bar{B}| = \frac{\mu_0}{4\pi} I \frac{\overbrace{dy}^{>0 \text{ if } y \text{ goes from } d \text{ to } l+d \text{ as shown}} \sin\theta}{r^2} \cdots (1) \quad (\text{here } dl = dy)$$

(again θ is angle between $d\bar{l}$ & $\bar{r}$)

Step $\boxed{II}$ Use symmetry : here we see that direction of $d\bar{B}$ is into page for all elements dy, i.e., irrespective of y (since $d\bar{l}$ is always upward vs. $\bar{r}$ points "to right") So, effectively we have a "scalar" addition (integral) of $d\bar{B}$'s to do here

Step $\boxed{III}$ Integrate $|d\bar{B}|$ from Eq. (1): note that the 3 variables entering $|d\bar{B}|$ above, i.e., r, θ and y are not all independent : we choose to write y and r in terms of θ as follows

$$y = R/\tan\theta \cdots (2) \quad \& \quad r^2 = y^2 + R^2 = R^2\left(\frac{1}{\tan^2\theta} + 1\right)$$
$$= R^2/\sin^2\theta \cdots (3)$$

Now, Eq. (2) gives
$$\frac{dy}{d\theta} = -\frac{R}{\tan^2\theta}\frac{d\tan\theta}{d\theta} = -\frac{R}{\tan^2\theta}\frac{1}{\cos^2\theta}$$
$$= -R/\sin^2\theta \cdots (5)$$

Plugging Eqs. (3) & (5) in Eq. (1) and integrating from θ_1 (corresponding to $y=d$) to θ_2 (i.e., $y = d+\ell$) gives

$$(\text{net})\ B\ (\text{again into page}) = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \overbrace{\left(\frac{-R}{\sin^2\theta}\,d\theta\right)}^{dy} \times \frac{\sin\theta}{\underbrace{\left\{R^2/\sin^2\theta\right.}_{r^2}}$$

$$= -\frac{\mu_0 I}{4\pi R}\int_{\theta_1}^{\theta_2}\sin\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi R}\left(\cos\theta\right)\Big|_{\theta_1}^{\theta_2} = \frac{\mu_0 I}{4\pi R}\left(\cos\theta_2 - \cos\theta_1\right) \quad \ldots (6)$$

Finally, we can relate $\theta_{1,2}$ to d, ℓ, R using $\tan\theta_1 = R/d$ and $\tan\theta_2 = R/(d+\ell)$

so that
$$\cos\theta_{1,2} = 1/\sqrt{(1+\tan^2\theta_{1,2})} = \frac{1}{\sqrt{1+\frac{R^2}{d^2 \text{ or } (d+\ell)^2}}}$$

$$= \frac{d \text{ or } (d+\ell)}{\sqrt{R^2 + d^2 \text{ or } (d+\ell)^2}} \quad \ldots (7)$$

Plugging Eq. (7) into Eq. (6) gives

$$\boxed{\bar{B} \text{ at } P \text{ (into page)} = \frac{\mu_0}{4\pi I R}\left(\frac{d+\ell}{\sqrt{R^2+(d+\ell)^2}} - \frac{d}{\sqrt{R^2+d^2}}\right)}$$