PHY 272: FIELDS
PROBLEM SET 3
due February 18, **before** *class*

A. Surface integrals

Continuing with our discussion of vector analysis through homework problems, this week we will talk about surface integrals. Imagine we have a surface S and a vector field $\mathbf{v}(\mathbf{r})$ in three dimensional space. The surface defines a unit vector orthogonal to it at each point in S. Well, to be precise, there is always a choice between a normal vector $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$, pointing on opposite directions. We will assume we chose one of them in a continuous way throughout the surface. This choice is called an "orientation" of the surface. (Also, there are some sick surfaces where we cannot choose an orientation over the whole surface, google images for "Möbius strip" or "Klein bottle" if interested). Now, we can break up the surface into infinitesimal little planes of area da, compute $\hat{\mathbf{n}}.\mathbf{v}$ at each one of them and sum them up

$$\int_S \hat{\mathbf{n}}.\mathbf{v}\ da. \tag{1}$$

This integral is know as the "surface integral of $\mathbf{v}$ over S (with orientation given by $\hat{\mathbf{n}}$)" or simply the "flux" of $\mathbf{v}$ through S. It is important to get some intuition of what the flux is. Imagine, for instance, that $\mathbf{v}$ represents the velocity of the air at each point in space and the surface S is the surface of a window. Then $\int_S \hat{\mathbf{n}}.\mathbf{v}\ da$ is teh amount of air leaving the room through the window per unit of time (or entering, if we choose the normal vector $\hat{\mathbf{n}}$ to pint inwards).

90% of the surface integrals in PHY272 will be such that $\hat{\mathbf{n}}.\mathbf{v}$ is a constant over the surface. Then it is *really* easy to compute the flux

$$\int_S \hat{\mathbf{n}}.\mathbf{v}\ da = \hat{\mathbf{n}}.\mathbf{v} \underbrace{\int_S da}_{\text{area}} = \hat{\mathbf{n}}.\mathbf{v} \times \text{area of S}. \tag{2}$$

In other words, the surface integral reduces to a multiplication. Let me give you some examples:

i) compute the flux of $\mathbf{v} = 2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + \hat{\mathbf{z}}$ through a square on the xy plane with corners at $0, \hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{x}} + \hat{\mathbf{y}}$.

ii) compute the flux of $\mathbf{v} = (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})/(x^2 + y^2 + z^2)^{3/2}$ through a sphere of radius R, centered at the origin and oriented outwards. Could you have predicted the dependence of the flux on R using dimensional analysis alone? Hint: think, make a drawing but do no use calculus!

Since that we are at it, I might as well tell you how to compute the flux for the general case where the integrand is not a constant. First, we should parametrize the surface like we did with paths. The way to do this is to give a vector function of *two* variables $\mathbf{r}(u, v)$ (let us call the variables u and v) such that, when u and v are varied over their range of values the vector $\mathbf{r}(u, v)$ touches on every point of the surface once and only once. To learn how this works, make a drawing of the surfaces described by

iii) $\mathbf{r}(u, v) = u\ \hat{\mathbf{x}} + v\ \hat{\mathbf{y}} + 3\ \hat{\mathbf{z}},\ \ 0 < u, v < 1$

iv) $\mathbf{r}(u, v) = \cos u\ \hat{\mathbf{x}} + \sin u\ \hat{\mathbf{y}} + 2v\ \hat{\mathbf{z}},\ \ 0 < u < 2\pi, 0 < v < 1$

Now try to parametrize the surface in items v) and vi)

v) a disk of radius R, lying on the $x - y$ plane and centered at the origin

vi) a sphere of radius R centered at the origin

The point $\mathbf{r}(u, v)$ is on the surface but so is the point $\mathbf{r}(u + \delta u, v)$ and $\mathbf{r}(u, v + \delta v)$, where δu and δv are infinitesimal shifts in u and v. That means that their difference $\mathbf{r}(u + \delta u, v) - \mathbf{r}(u, v)$ and $\mathbf{r}(u, v + \delta v) - \mathbf{r}(u, v)$ are infinitesimal vectors *tangent* to the surface (you would be helped here by making a drawing). That means we can find a vector orthogonal to the surface by taking the cross product

$$\hat{\mathbf{n}} = \frac{\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}}{|\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}|} \tag{3}$$

where we also divide normalized the normal vector.

vii) compute the normalized unit vector of the surfaces in problems v) and vi).

The element of area da is given by $da = |\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}|\, dudv$ so the surface integral is given, after we choose a parametrization $\mathbf{r}(u,v)$ of the surface S, by

$$\int_{u_i}^{u_f} \int_{v_i}^{v_f} dudv \left(\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv} \right) .\mathbf{v}. \tag{4}$$

viii) use this formula to compute the flux of $\mathbf{v} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ over the disk in problem v) (oriented upwards).

Last week we learned that the gradient was the “opposite” to a line integral in the sense that a line integral of a gradient of the function gave the value of the function at the ends of the path. Here too, there is a kind of derivative that is the “opposite” to a surface integral. It’s called the “divergence” and we will learn all about it next week (maybe).

B. Charged cylinder

Find the electric field generated by an uniformly charged cylindrical surface of radius R, height h and total charge Q along its axis. Check that the field at a point very far away from the cylinder has the expected form.

C. Field of a bar, away from the center

Find the electric field generated by an uniformly charged thin bar of length L along a straight line orthogonal to the bar passing through one of its extremes.