RL Circuit Example

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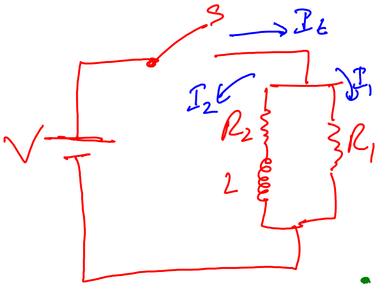
1:06 AM

Switch is closed at t=0 seconds, Find the current through the battery as a function of time:



@ $t=0$, Switch is closed.
Find the current through the battery as a function of time:

$V = I_1 R_1$, $V = I_2 R_2 + L \frac{dI_2}{dt}$ .... ①

or $I_t = I_1 + I_2$

∴ $I_t = \frac{V}{R_1} + I_2 \Rightarrow I_2 = I_t - \frac{V}{R_1}$ .... ②

Solve these 3 equations for $I_t(t)$.
Also, apply Boundary conditions @ $t \to 0$

∴ using equation ① & ②

$$V = \left(I_t - \frac{V}{R_1}\right) R_2 + L \frac{d}{dt}\left(I_t - \underbrace{\frac{V}{R_1}}_{\text{constant}}\right)$$

$$\frac{V}{R_2} = I_t - \frac{V}{R_1} + \frac{L}{R_2}\frac{d}{dt} I_t$$

The rest is math, solving for $I_t(t)$

$$\Rightarrow \frac{R_2}{L}\left[\frac{V}{R_2} - I_t + \frac{V}{R_1}\right] = \frac{d}{dt} I_t$$

$$\Rightarrow \underbrace{\frac{R_2}{L} V \left(\frac{1}{R_1} + \frac{1}{R_2}\right)}_{\equiv C \text{ a constant}} - \frac{R_2}{L} I_t = \frac{d}{dt} I_t$$

Let $-I' = C - \frac{R_2}{L} I_t \Rightarrow \frac{dI'}{dt} = \frac{R_2}{L}\frac{dI_t}{dt}$

$$\therefore -I' = \frac{L}{R_2}\frac{dI'}{dt} \Rightarrow -\frac{R_2}{L} dt = \frac{dI'}{I'}$$

integrate $\Rightarrow \ln I' \Big|_{I'_0}^{I'} = -\frac{R_2}{L} t$ Let $\boxed{\tau \equiv L/R_2}$

$$\Rightarrow I' = I'_0 e^{-t/\tau}$$

no current through inductor @ $t=0$

Boundary Conditions : @ $t=0$ $I_t = I_1 = \frac{V}{R_1} \Rightarrow I'_0 = -C + \frac{R_2}{L}\frac{V}{R_1}$

$$\therefore -C + \frac{I_t}{\tau} = \left(-C + \frac{R_2}{R_1}\frac{V}{L}\right) e^{-t/\tau}$$

$$I_t(t) = \tau C \left(1 - e^{-t/\tau}\right) + \frac{R_2 V}{L}\frac{\tau}{R_1} e^{-t/\tau} \; ; \tau = \frac{L}{R_2}, \; C = \frac{R_2 V}{L}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$I_t(t) = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(1 - e^{-t/\tau}\right) + \frac{V}{R_1} e^{-t/\tau}$$

$$\boxed{I_t(t) = \frac{V}{R_1} + \frac{V}{R_2}\left(1 - e^{-t/\tau}\right)}$$

@ $t=0$, $I_t(t) = V/R_1$

@ $t \to \infty$, $I_t(t) = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

This makes sense because
as $t \to \infty$, current is steady & inductor can be ignored

$\Rightarrow I_t = \frac{V}{R_{eq}}$, $\frac{1}{R_{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

$\Rightarrow I_t = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$