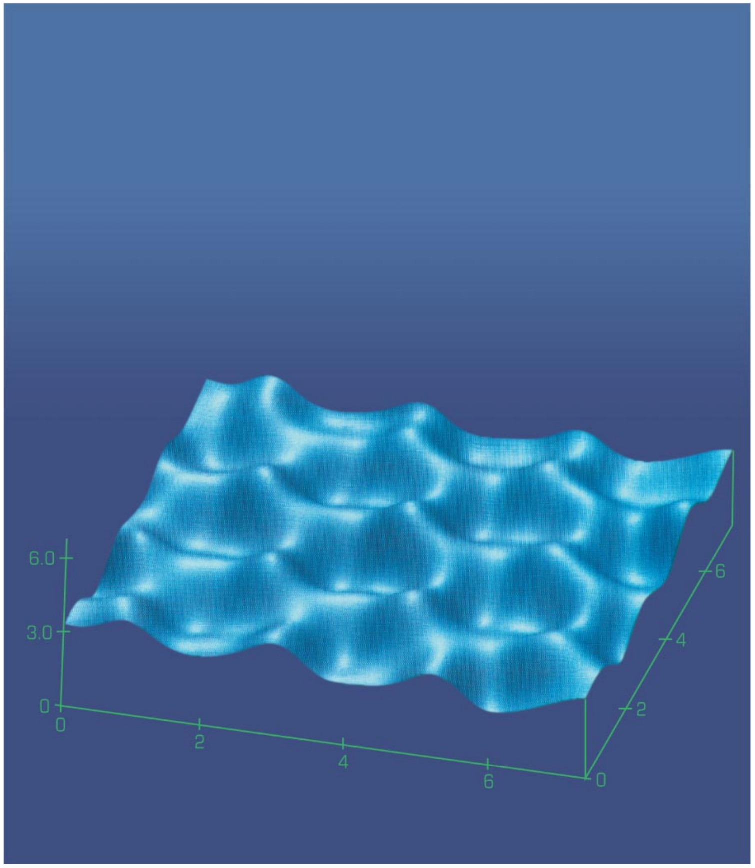
Chapter 40. Wave Functions and Uncertainty

The wave function characterizes particles in terms of the probability of finding them at various points in space. This scanning tunneling microscope image of graphite shows the most probable place to find electrons.



Chapter Goal: To introduce the wave-function description of matter and learn how it is interpreted.

Student Learning Objectives

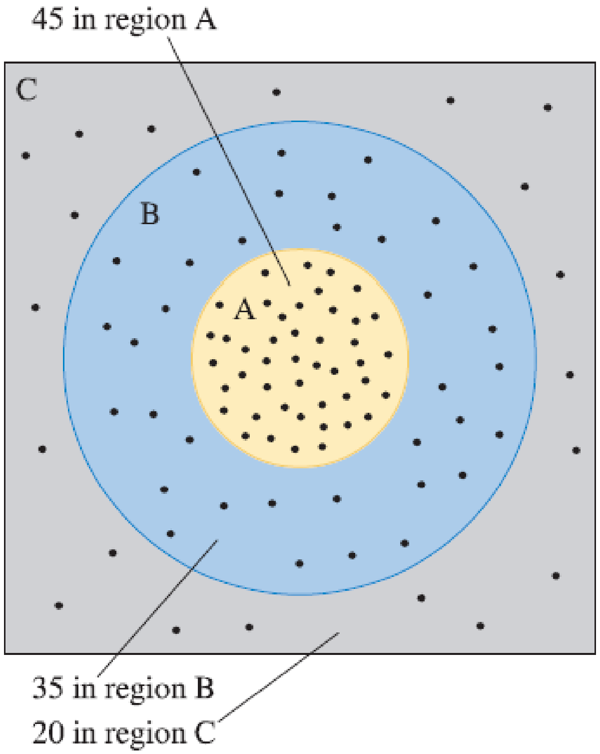
- To introduce the *wave function* as the descriptor of particles in quantum mechanics.
- To provide the wave function with a probabilistic interpretation.
- To understand the wave function through pictorial and graphical exercises.
- To introduce the idea of normalization.
- To recognize the limitations on knowledge imposed by the Heisenberg uncertainty principle.

Chapter 40. Wave Functions and Uncertainty

Topics:

- Waves, Particles, and the Double-Slit Experiment
- Connecting the Wave and Photon Views
- The Wave Function
- Normalization
- Wave Packets
- The Heisenberg Uncertainty Principle

FIGURE 40.2 One hundred throws at a dart board.



$$P_A = \lim_{N_{tot}\to\infty} \frac{N_A}{N_{tot}}$$

$$P_A \approx 45\%,\ P_B \approx 35\%, \text{ and } P_C \approx 20\%$$

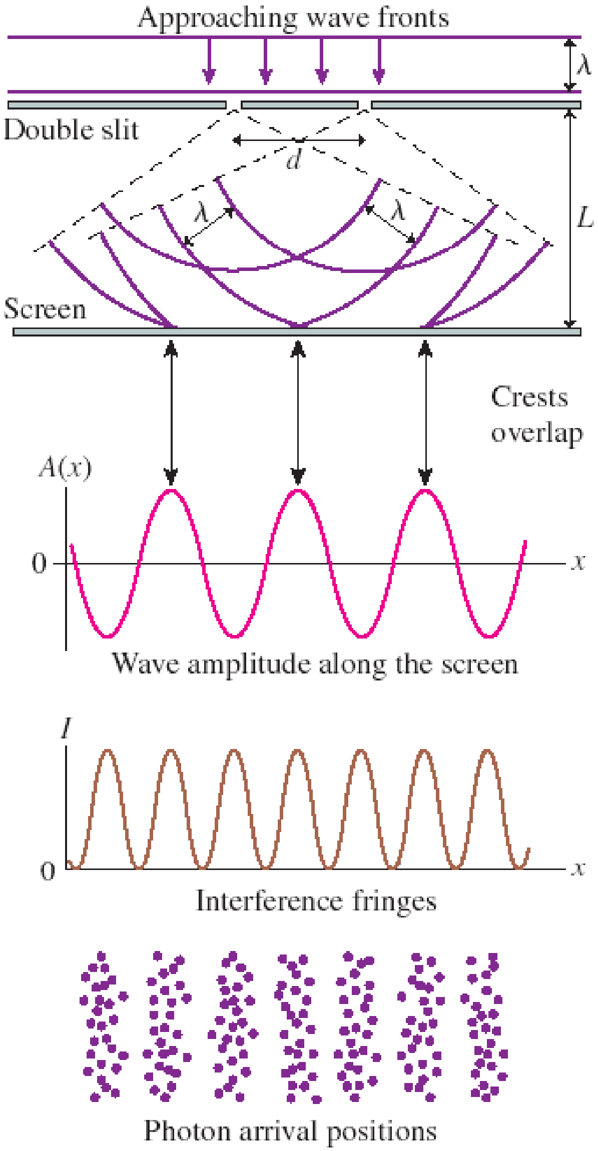
$$P_{A\text{ or }B} = \lim_{N_{tot}\to\infty} \frac{N_{A\text{ or }B}}{N_{tot}} = \lim_{N_{tot}\to\infty} \frac{N_A + N_B}{N_{tot}}$$

$$= \lim_{N_{tot}\to\infty} \frac{N_A}{N_{tot}} + \lim_{N_{tot}\to\infty} \frac{N_B}{N_{tot}} = P_A + P_B$$

$$P_{\text{somewhere}} = P_{A\text{ or }B\text{ or }C} = P_A + P_B + P_C = 0.45 + 0.35 + 0.20 = 1.00$$

If you throw N darts, **expected value** $N_{A\text{ expected}} = NP_A$

FIGURE 40.1 The double-slit experiment with light.



Review double slit

In general: (assuming E_1 + E_2 linearly polarized in same plane)

$$E_1 = E_0 \cos(kx_1 - wt + \phi_{10}) \quad , \quad E_2 = E_0 \cos(kx_2 - wt + \phi_{20})$$

$$= E_0\, e^{i(kx_1 - wt + \phi_{10})} \qquad\qquad = E_0\, e^{i(kx_2 - wt + \phi_{20})}$$

Provided we take the real part after summation since:

$$\mathrm{Re}(e^{i\theta}) = \mathrm{Re}(\cos\theta + i\sin\theta) = \cos\theta$$

$$E_t = E_1 + E_2 = E_0 \left[e^{ikx_1} e^{-iwt} e^{i\phi_{10}} + e^{ikx_2} e^{-iwt} e^{i\phi_{20}} \right]$$

$$= E_0\, e^{-iwt} \left[e^{ikx_1} e^{i\phi_{10}} + e^{ikx_2} e^{i\phi_{20}} \right]$$

$$= E_0\, e^{-iwt} e^{ik\frac{x_1}{2}} e^{ik\frac{x_2}{2}} e^{i\frac{\phi_{10}}{2}} e^{i\frac{\phi_{20}}{2}} \left[e^{ik\frac{x_1}{2}} e^{-ik\frac{x_2}{2}} e^{i\frac{\phi_{10}}{2}} e^{-i\frac{\phi_{20}}{2}} + e^{ikx_2/2} e^{-ikx_1/2} e^{i\phi_{20}/2} e^{-i\phi_{10}/2} \right]$$

$$= E_0\, e^{i(k x_{avg} - wt + \phi_{avg})} \left[\underbrace{e^{-i(k\frac{\Delta x}{2} + \frac{\Delta\phi_0}{2})} + e^{i(k\Delta x/2 + \Delta\phi_0/2)}}_{2\cos\left(\frac{k\Delta x + \Delta\phi_0}{2}\right)} \right]$$

Note: $e^{i\alpha} + e^{-i\alpha} =$

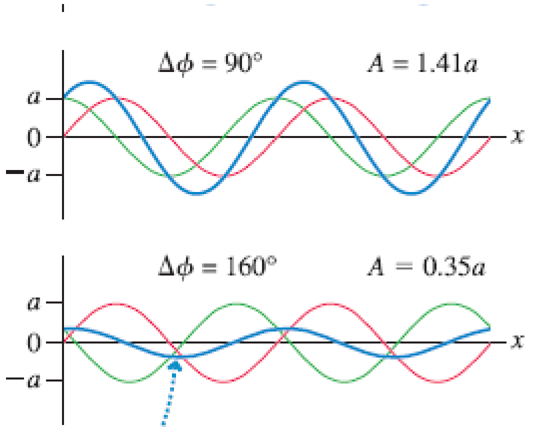
$(\cos\alpha + i\sin\alpha) + (\cos\alpha - i\sin\alpha)$

$= 2\cos\alpha$

Take Real part:

$$\boxed{E_t = 2E_0 \cos\left(\frac{\overset{\Delta OPL}{k\Delta x} + \Delta\phi_0}{2}\right) \cos(k x_{avg} - wt + \phi_{avg})}$$

$$E_t = \underbrace{2E_0 \cos\underbrace{\left(\frac{k\overbrace{\Delta x}^{\Delta OPL} + \Delta\phi_0}{2}\right)}_{\equiv \Delta\phi/2}}_{\text{Amplitude, } A} \cos\left(k x_{avg} - \omega t + \phi_{avg}\right)$$



For Δϕ = 160°, the interference is destructive but not perfect destructive.

$$A = \left|2E_0 \cos\frac{\Delta\phi}{2}\right|$$

Constructive interference (amplitude @ maximum $\Rightarrow \pm 2E_0$)

$$\Rightarrow \cos\frac{\Delta\phi}{2} = \pm 1 \Rightarrow \frac{\Delta\phi}{2} = m\pi \Rightarrow \Delta\phi = 2m\pi \quad , \; m = 0, \pm 1, \pm 2, \ldots$$

If sources are in phase $(\Delta\phi_0 = 0)$ $\Rightarrow \frac{\Delta\phi}{2} = \frac{k\Delta x}{2} = \pi\frac{\Delta x}{\lambda} = m\pi$

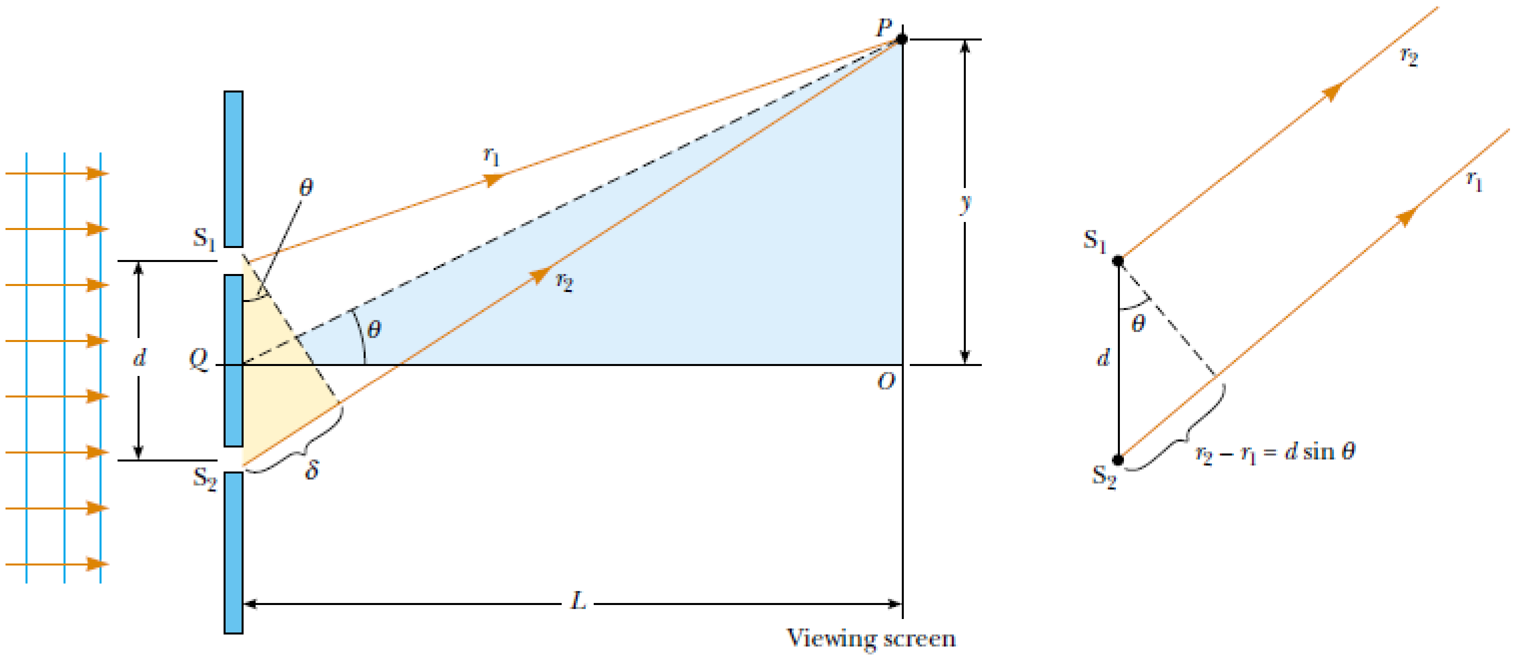
or $\Delta x = \Delta OPL = m\lambda$

Destructive interference (amplitude = 0)

$$\Rightarrow \cos\frac{\Delta\phi}{2} = 0 \Rightarrow \frac{\Delta\phi}{2} = \left(\frac{2m+1}{2}\right)\pi \Rightarrow \Delta\phi = (2m+1)\pi$$

If source are in phase $(\Delta\phi_0 = 0)$ $\Rightarrow \Delta\phi = \frac{k\Delta x}{2} = \pi\frac{\Delta x}{\lambda} = \frac{(2m+1)}{2}\pi$

or $\Delta x = \Delta OPL = \left(\frac{2m+1}{2}\right)\lambda$



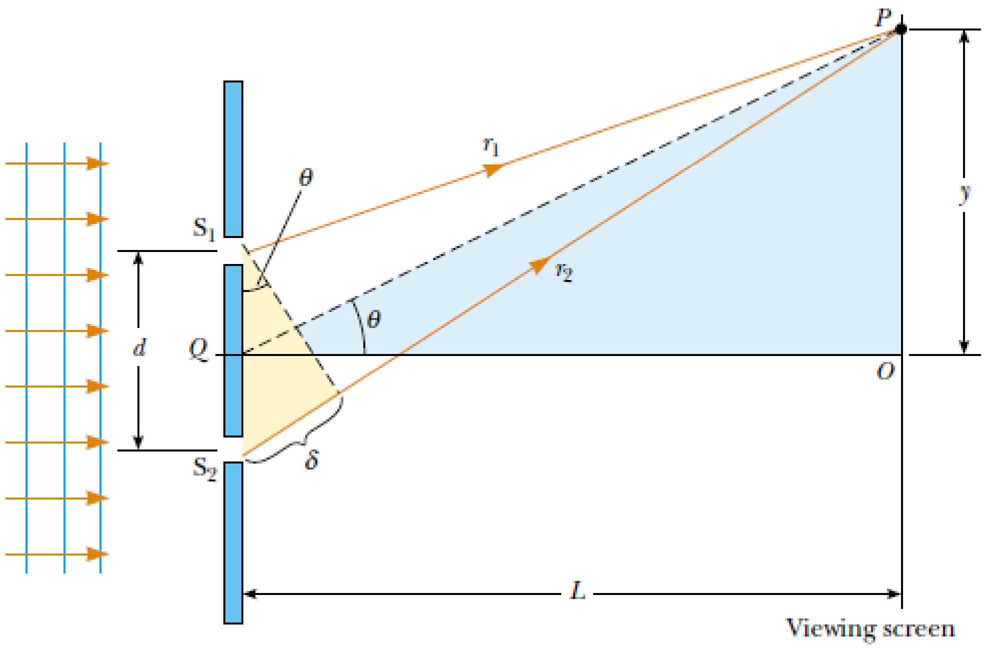
$$E = 2E_0 \cos\left(\frac{\Delta\phi}{2}\right) \cos\left(k x_{avg} - \omega t + \phi_{avg}\right)$$

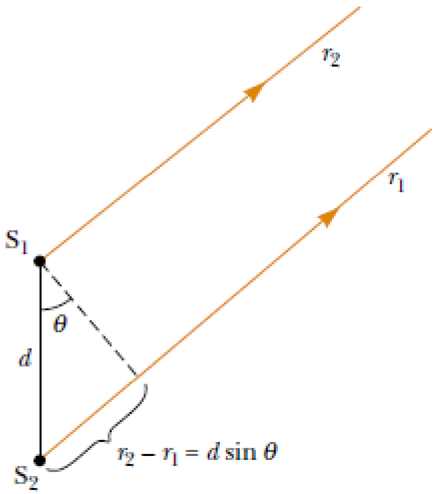
$$\delta = d \sin\theta = \Delta OPL$$

$$\Delta\phi = k\delta = \frac{2\pi}{\lambda} d \sin\theta$$

for $L >> d + y$, $\sin\theta \approx \theta = \frac{y}{L}$

$$\Rightarrow \frac{\Delta\phi}{2} \approx \frac{\pi d}{\lambda} \frac{y}{L}$$



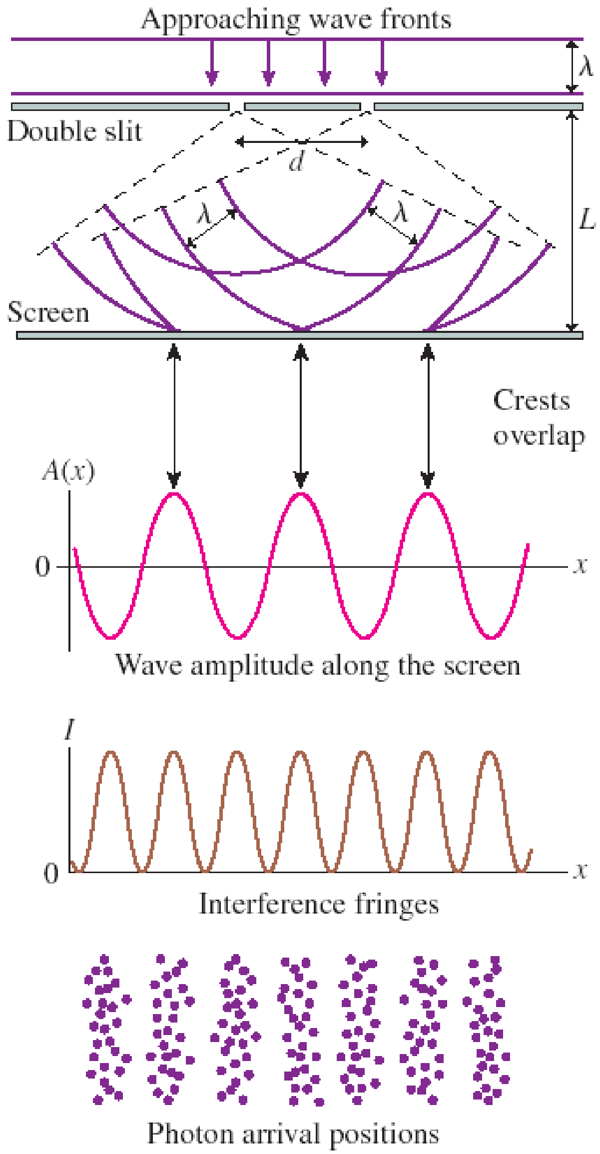


$$\therefore E = \underbrace{2E_0 \cos\left(\frac{\pi d}{\lambda}\frac{y}{L}\right)}_{A(y),\ \text{"Amplitude Function"}} \cos(k x_{avg} - \omega t + \phi_{avg})$$

$$I \propto |E|^2 \quad \therefore \quad I = |A(y)|^2 \cos^2(k x_{avg} - \omega t + \phi_{avg})$$

$$I_{avg} \propto |A(y)|^2$$

FIGURE 40.1 The double-slit experiment with light.



$$A(x) = 2E_0 \cos\left(\frac{\pi d}{\lambda} \; \frac{x}{2}\right)$$

$$I_{avg}(x) \propto |A(x)|^2$$

$$\text{Prob}(\text{in } \delta x \text{ at } x) \propto |A(x)|^2 \delta x$$

FIGURE 40.1 The double-slit experiment with light.

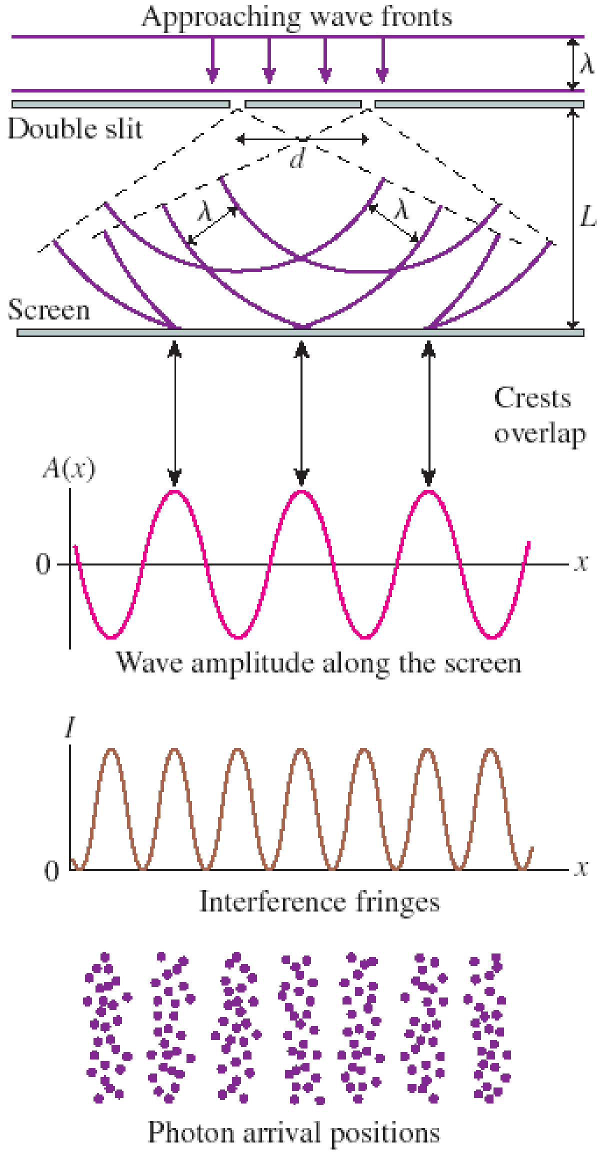
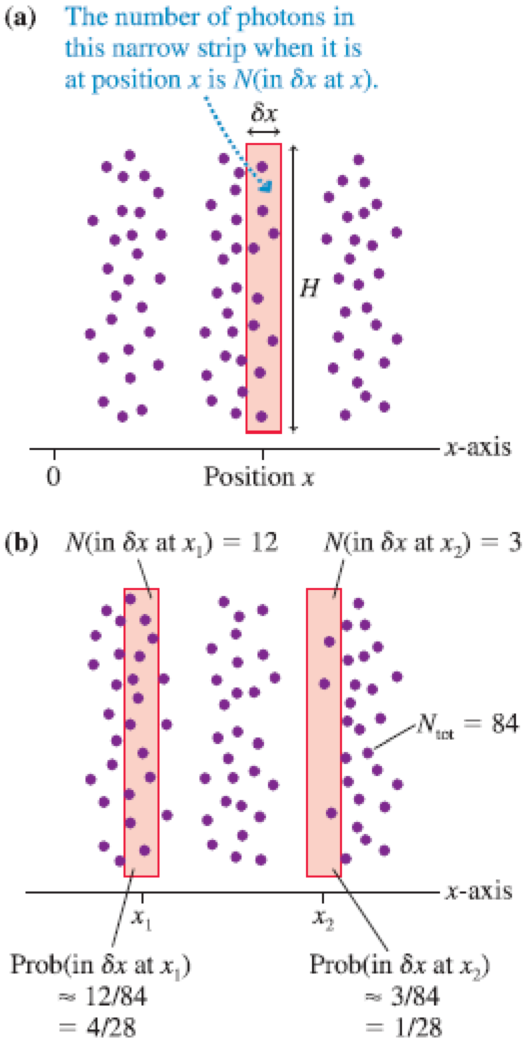


FIGURE 40.3 A strip of width δx at position x.



$$\text{Prob}(\text{in } \delta x \text{ at } x) \propto |A(x)|^2 \delta x$$

Connecting the Wave and Photon Views

The *intensity of the light wave* is correlated with the *probability of detecting photons.* That is, photons are more likely to be detected at those points where the wave intensity is high and less likely to be detected at those points where the wave intensity is low.

The probability of detecting a photon at a particular point is directly proportional to the square of the light-wave amplitude function at that point:

$$\text{Prob}(\text{in } \delta x \text{ at } x) \propto |A(x)|^2 \delta x$$

Probability Density

We can define the probability density *P*(*x*) such that

$$\text{Prob}(\text{in } \delta x \text{ at } x) = P(x)\,\delta x$$

In one dimension, probability density has SI units of m^{-1}. Thus the probability density multiplied by a length yields a dimensionless probability.

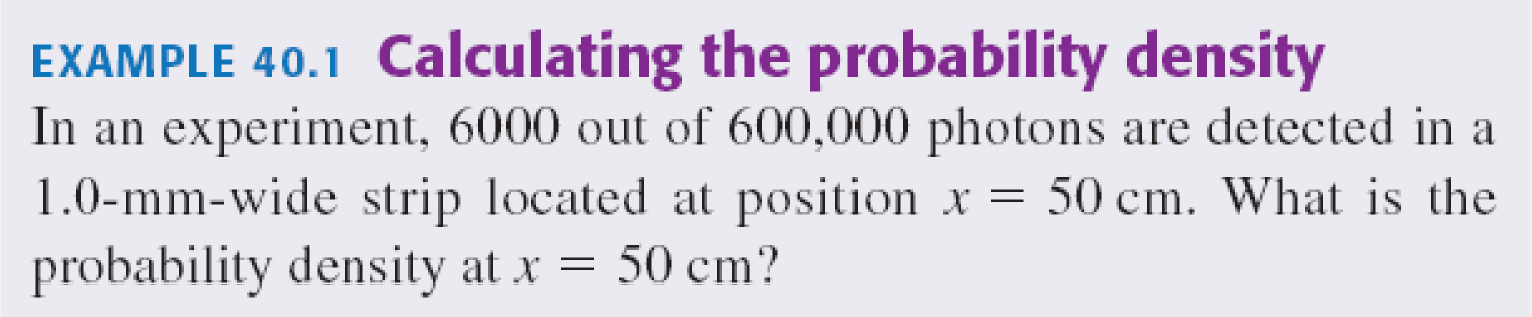
NOTE: *P*(*x*) itself is *not* a probability. You must multiply the probability density by a length to find an actual probability.

The photon probability density is directly proportional to the square of the light-wave amplitude:

$$P(x) \propto |A(x)|^2$$

EXAMPLE 40.1 Calculating the probability density

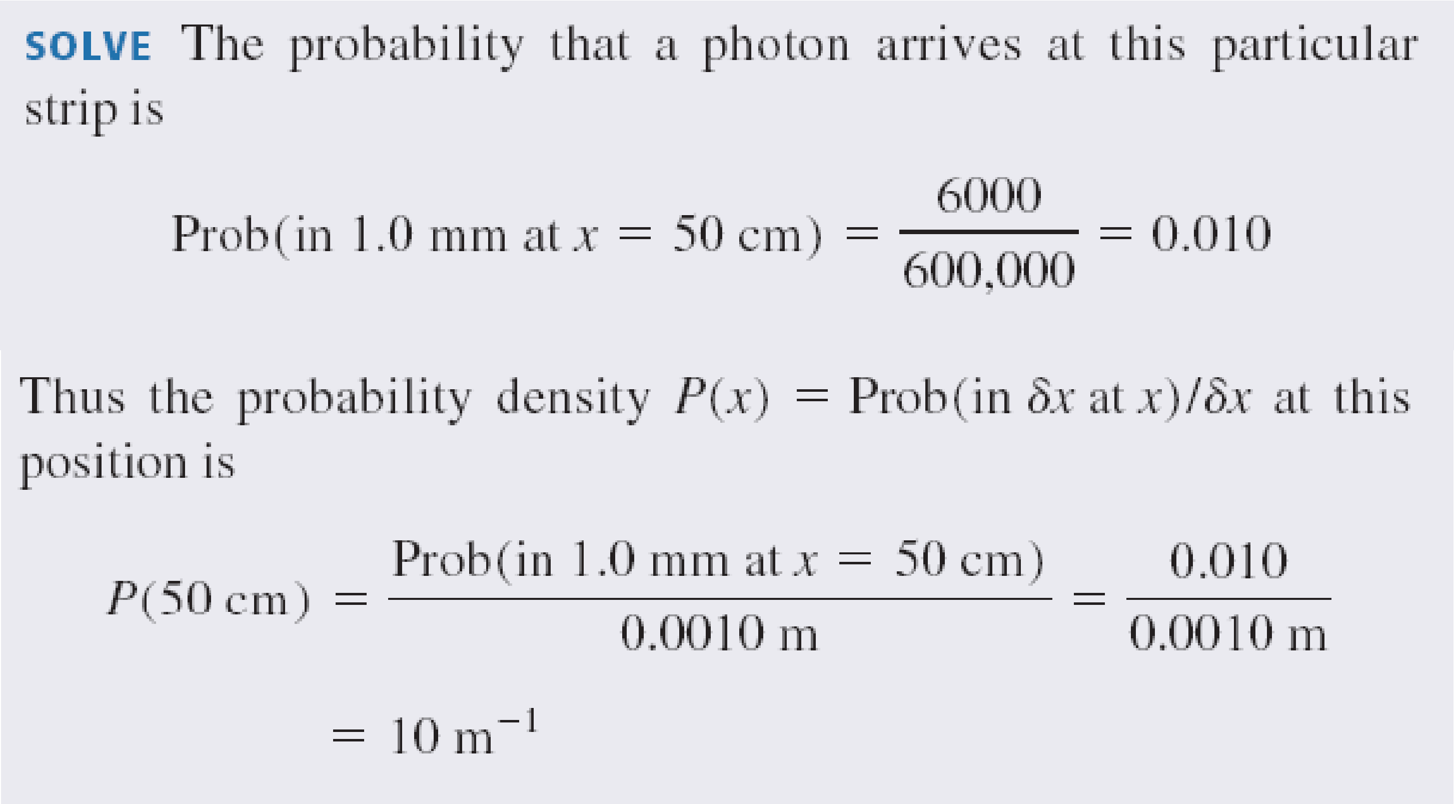
QUESTION:



EXAMPLE 40.1 **Calculating the probability density**

In an experiment, 6000 out of 600,000 photons are detected in a 1.0-mm-wide strip located at position $x = 50$ cm. What is the probability density at $x = 50$ cm?

EXAMPLE 40.1 Calculating the probability density



SOLVE The probability that a photon arrives at this particular strip is

$$\text{Prob(in 1.0 mm at } x = 50\text{ cm)} = \frac{6000}{600{,}000} = 0.010$$

Thus the probability density $P(x) = \text{Prob(in } \delta x \text{ at } x)/\delta x$ at this position is

$$P(50\text{ cm}) = \frac{\text{Prob(in 1.0 mm at } x = 50\text{ cm)}}{0.0010\text{ m}} = \frac{0.010}{0.0010\text{ m}}$$

$$= 10\text{ m}^{-1}$$

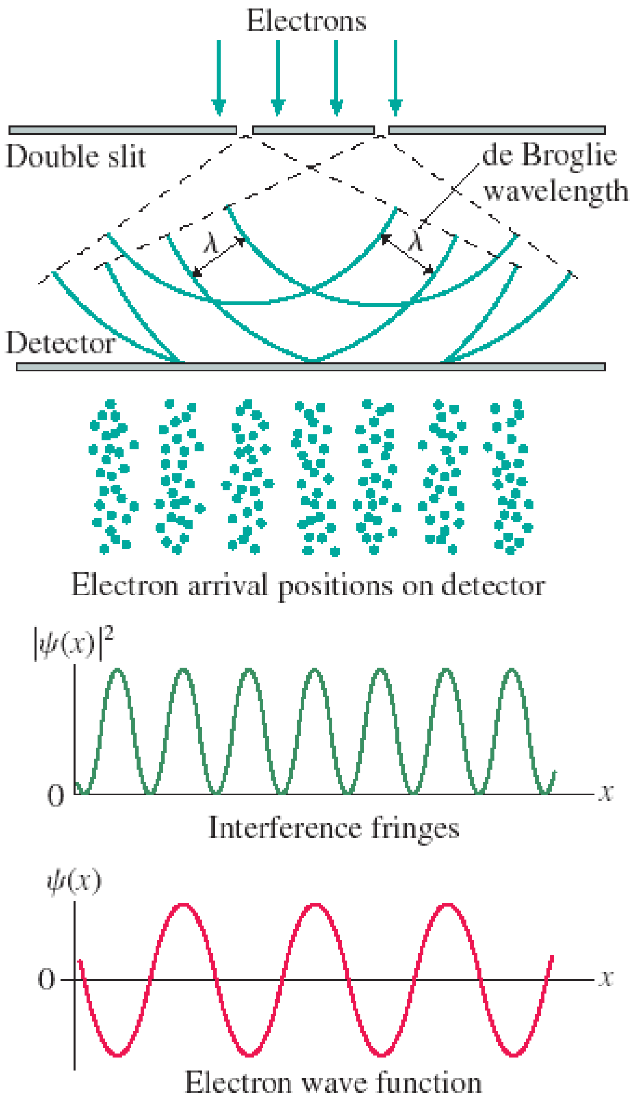
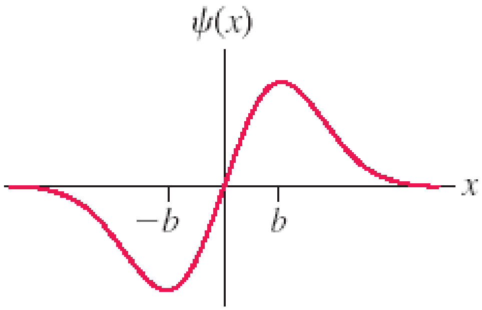


FIGURE 40.5 The double-slit experiment with electrons.

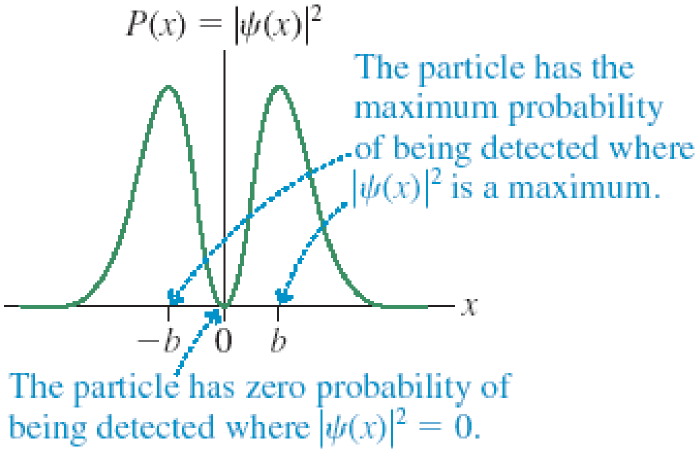
- Electrons behave in similar way: probability density looks just like photons
- Postulate $\exists\ \Psi(x)$ for matter waves similar to $A(x)$ for photons — Note that $A(x)$ is the Electric field Amplitude, $\Psi(x)$ is something else!
- Probability density $\propto |\Psi(x)|^2$ just like photons where $P(x) \propto |A(x)|^2 dx$

FIGURE 40.6 The square of the wave function is the probability density for detecting the electron at various values of the position x.

(a) Wave function



(b) Probability density



The particle has the maximum probability of being detected where $|\psi(x)|^2$ is a maximum.

The particle has zero probability of being detected where $|\psi(x)|^2 = 0$.

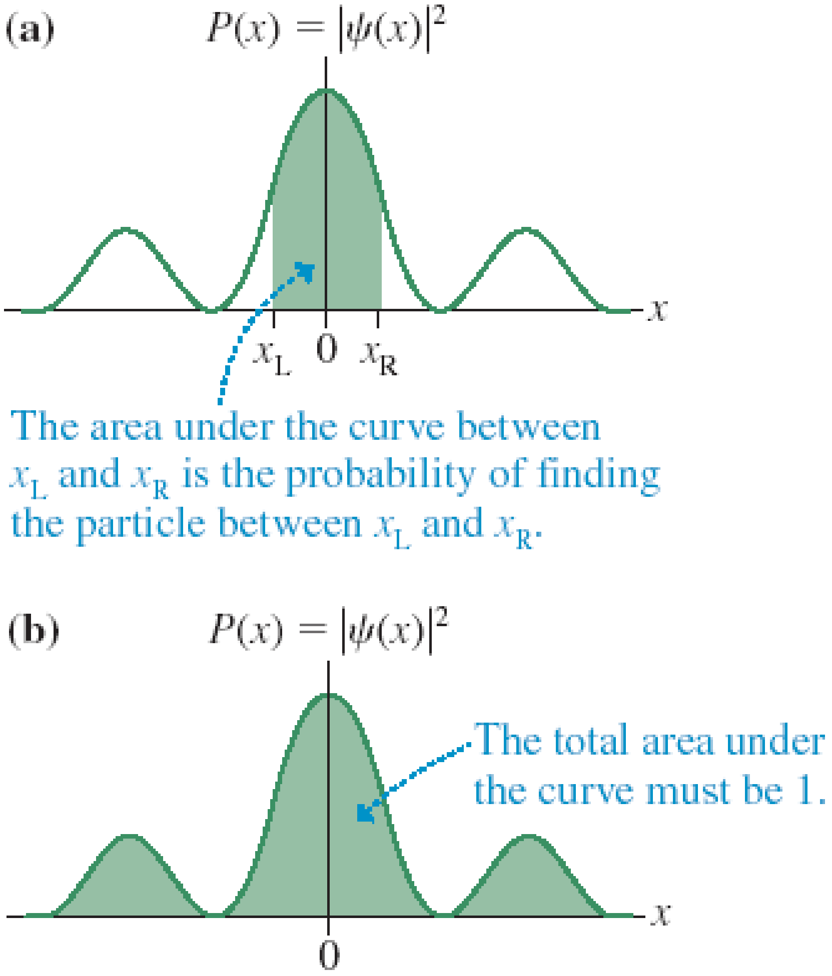
Normalization

- A photon or electron has to land *somewhere* on the detector after passing through an experimental apparatus.
- Consequently, the probability that it will be detected at *some* position is 100%.
- The statement that the photon or electron has to land *somewhere* on the *x*-axis is expressed mathematically as

$$\int_{-\infty} P(x)\,dx = \int_{-\infty} |\psi(x)|^2 dx = 1$$

- Any wave function must satisfy this **normalization condition.**

FIGURE 40.8 The area under the probability density curve is a probability.

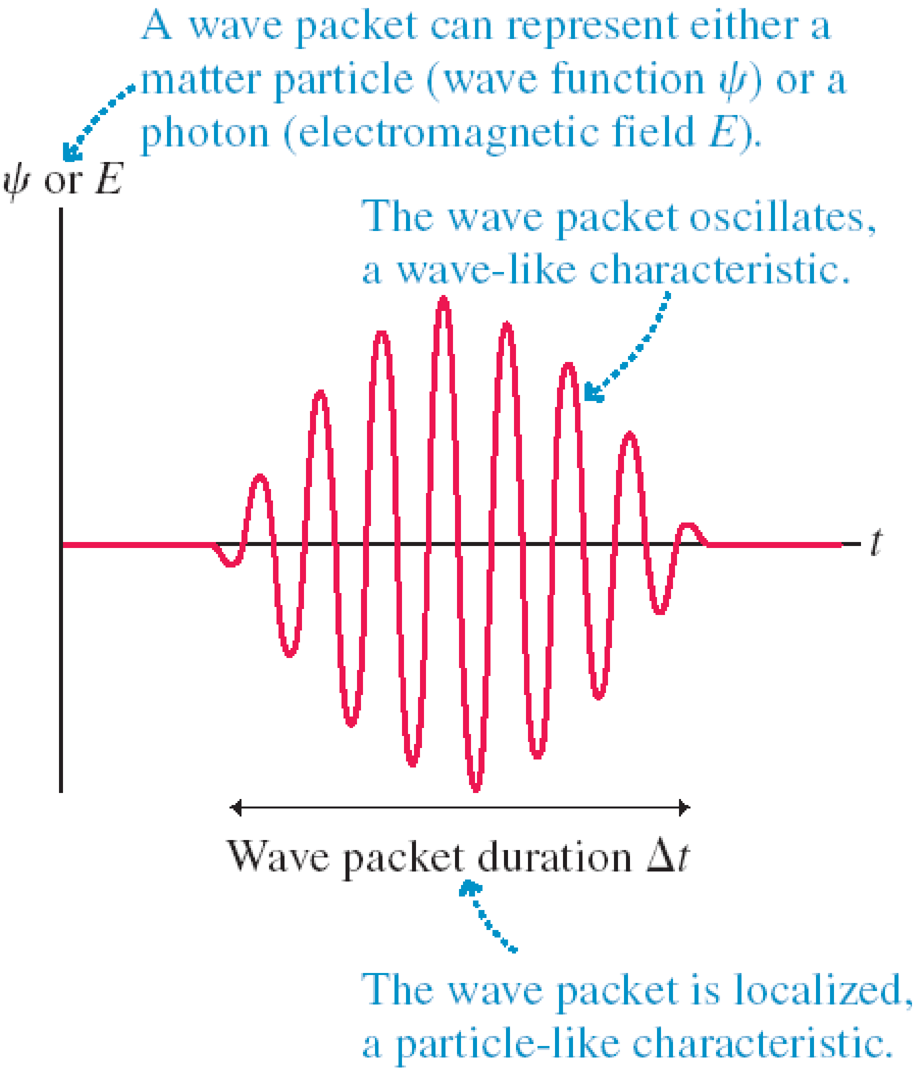


The area under the curve between x_L and x_R is the probability of finding the particle between x_L and x_R.

The total area under the curve must be 1.

Wave Packets

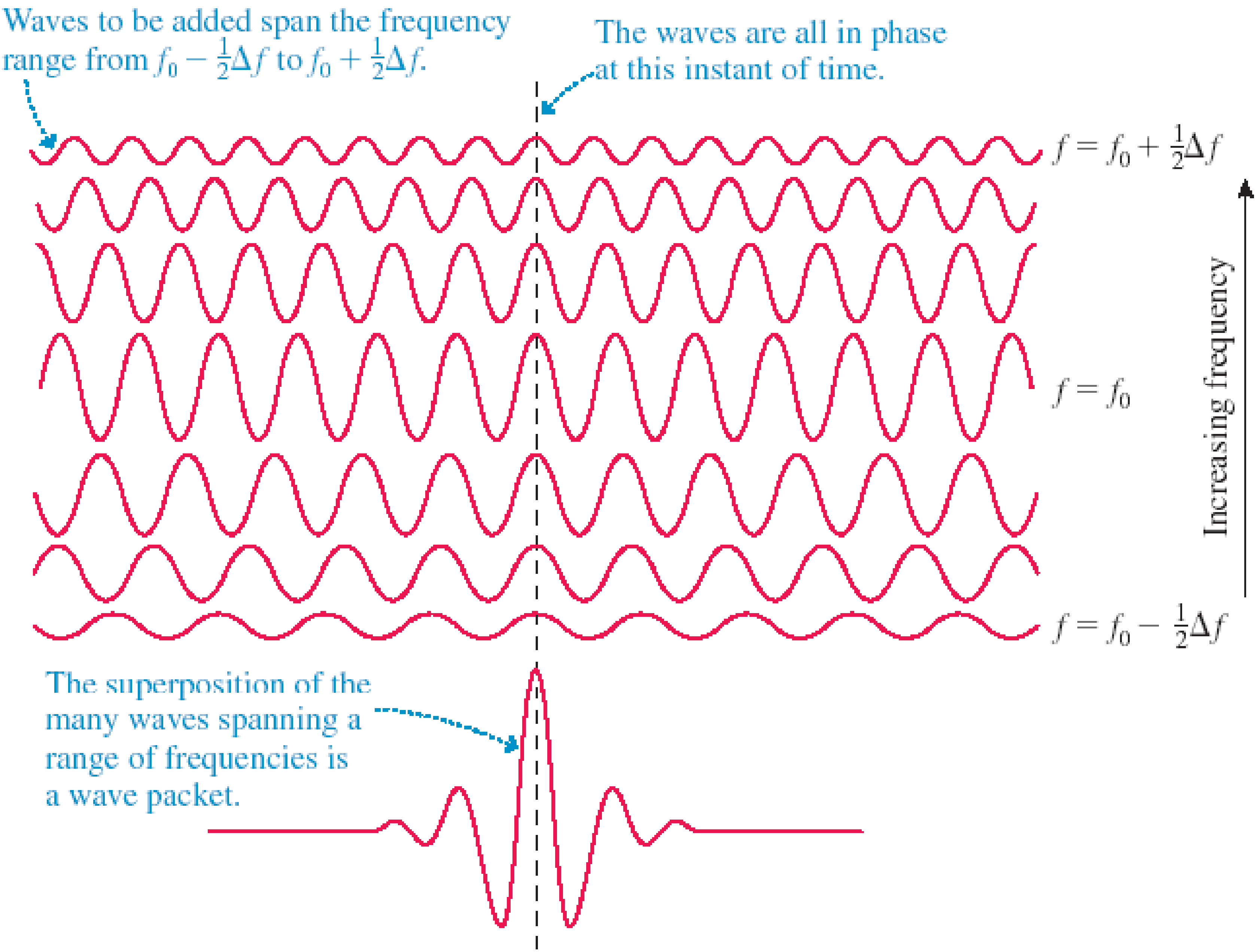
FIGURE 40.12 History graph of a wave packet with duration Δt.



A wave packet can represent either a matter particle (wave function ψ) or a photon (electromagnetic field E).

The wave packet oscillates, a wave-like characteristic.

The wave packet is localized, a particle-like characteristic.



Waves to be added span the frequency range from $f_0 - \frac{1}{2}\Delta f$ to $f_0 + \frac{1}{2}\Delta f$.

The waves are all in phase at this instant of time.

The superposition of the many waves spanning a range of frequencies is a wave packet.

Wave Packets

Suppose a single nonrepeating wave packet of duration Δ*t* is created by the superposition of *many* waves that span a range of frequencies Δ*f*.

Fourier analysis shows that for *any* wave packet

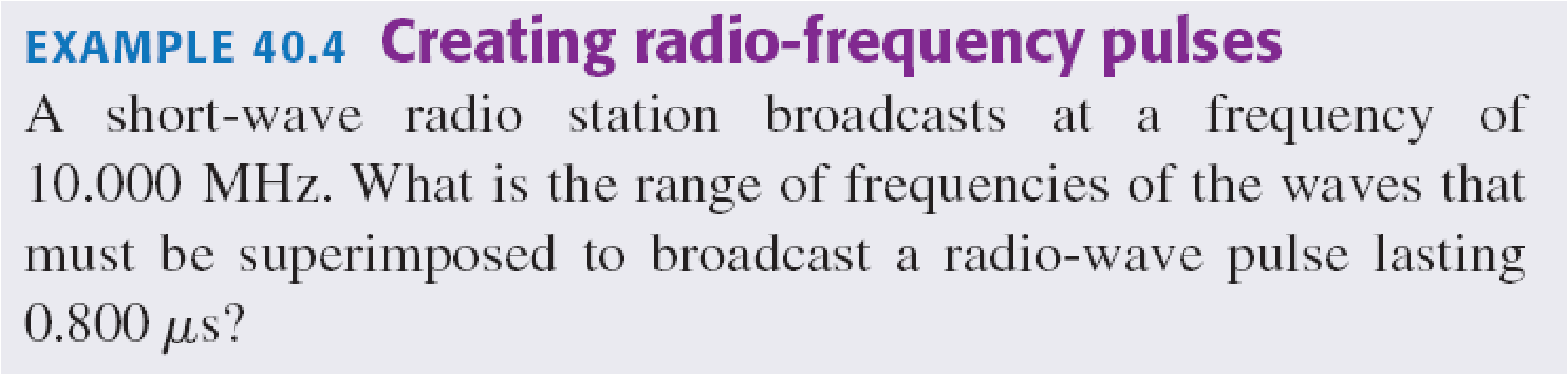
$$\Delta f \Delta t \approx 1$$

We have not given a precise definition of Δ*t* and Δ*f* for a general wave packet.

The quantity Δ*t* is "about how long the wave packet lasts," while Δ*f* is "about the range of frequencies needing to be superimposed to produce this wave packet."

EXAMPLE 40.4 Creating radio-frequency pulses

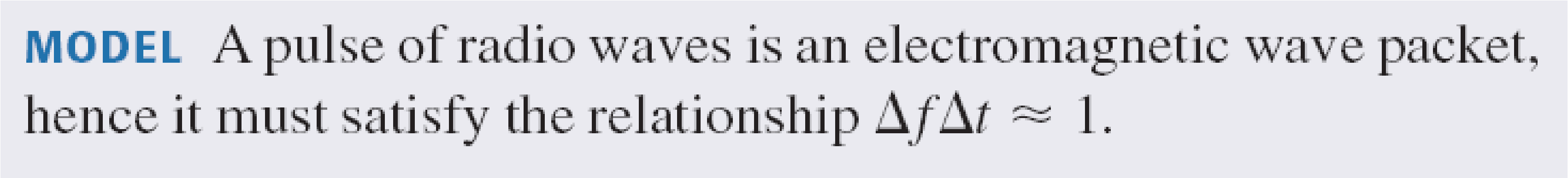
QUESTION:



EXAMPLE 40.4 **Creating radio-frequency pulses**

A short-wave radio station broadcasts at a frequency of 10.000 MHz. What is the range of frequencies of the waves that must be superimposed to broadcast a radio-wave pulse lasting 0.800 μs?

EXAMPLE 40.4 Creating radio-frequency pulses

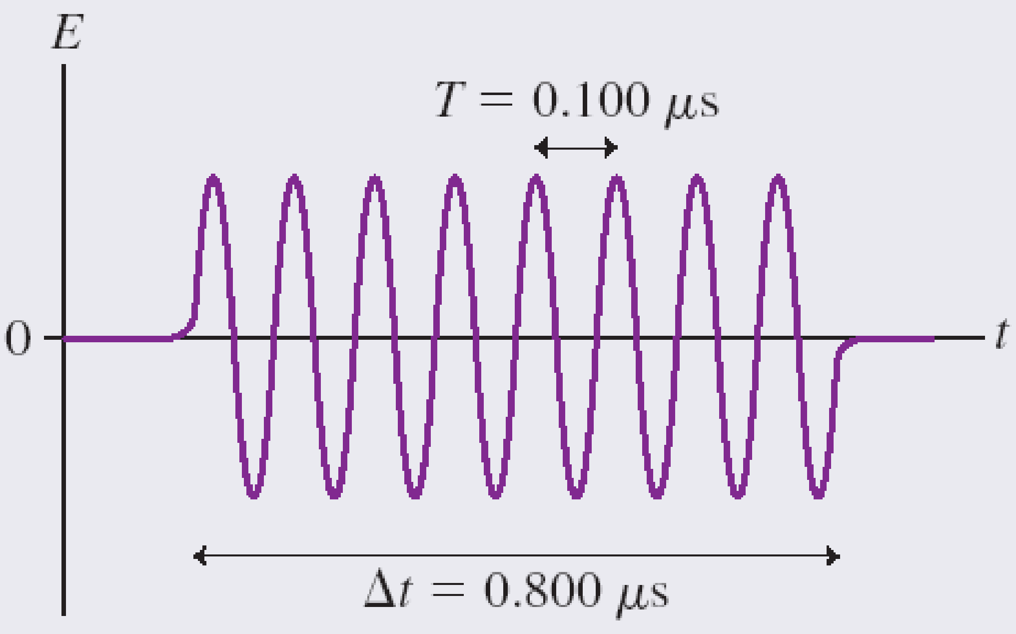


MODEL A pulse of radio waves is an electromagnetic wave packet, hence it must satisfy the relationship $\Delta f \Delta t \approx 1$.

EXAMPLE 40.4 Creating radio-frequency pulses

VISUALIZE **FIGURE 40.15** shows the pulse.

FIGURE 40.15 A pulse of radio waves.



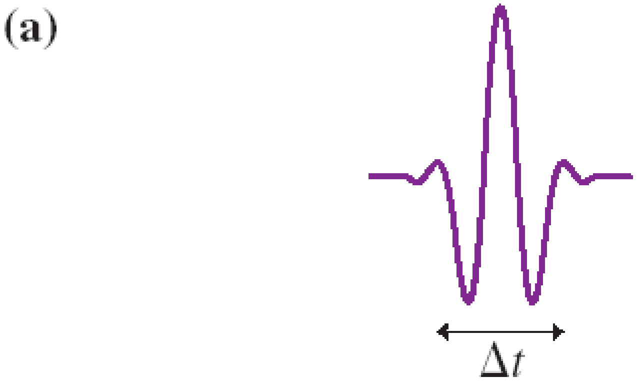
SOLVE The period of a 10.000 MHz oscillation is 0.100 μs. A pulse 0.800 μs in duration is 8 oscillations of the wave. Although the station broadcasts at a nominal frequency of 10.000 MHz, this pulse is not a pure 10.000 MHz oscillation. Instead, the pulse has been created by the superposition of many waves whose frequencies span

$$\Delta f \approx \frac{1}{\Delta t} = \frac{1}{0.800 \times 10^{-6}\ \mathrm{s}} = 1.250 \times 10^{6}\ \mathrm{Hz} = 1.250\ \mathrm{MHz}$$

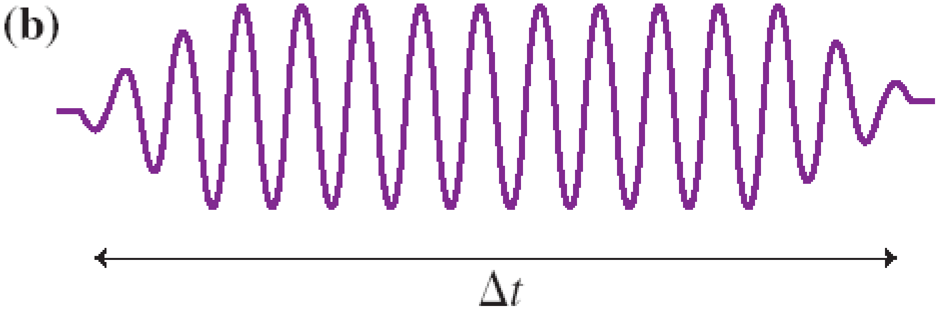
This range of frequencies will be centered at the 10.000 MHz broadcast frequency, so the waves that must be superimposed to create this pulse span the frequency range

$$9.375\ \mathrm{MHz} \leq f \leq 10.625\ \mathrm{MHz}$$

FIGURE 40.16 Two wave packets with different Δt.



This wave packet has a large frequency uncertainty Δf.



This wave packet has a small frequency uncertainty Δf.

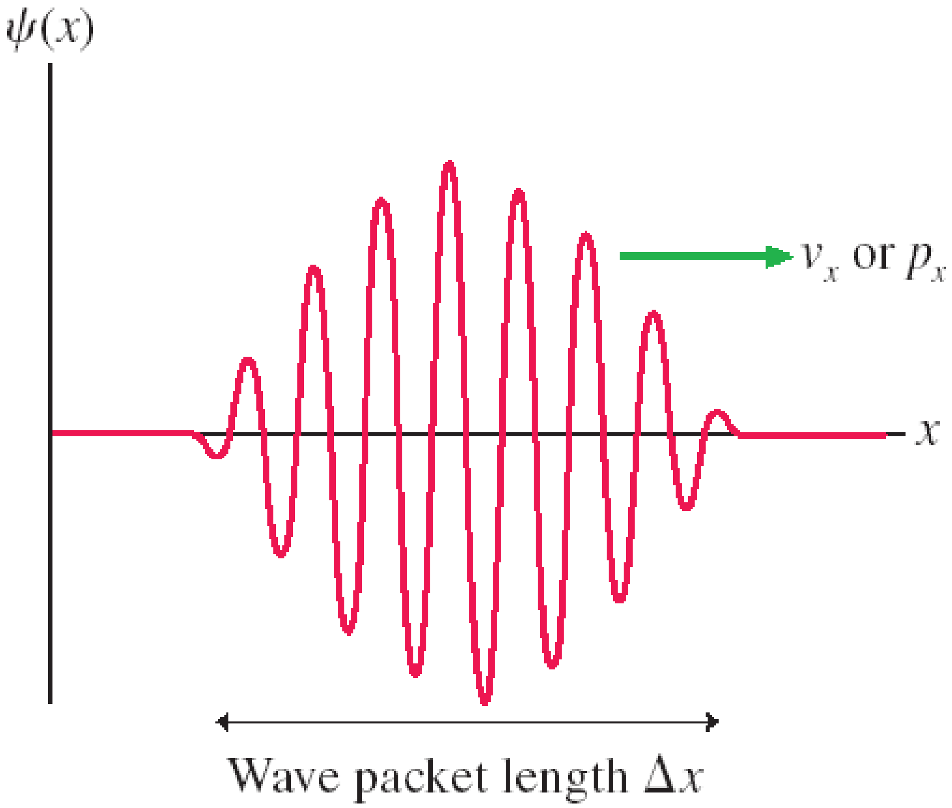
The Heisenberg Uncertainty Principle

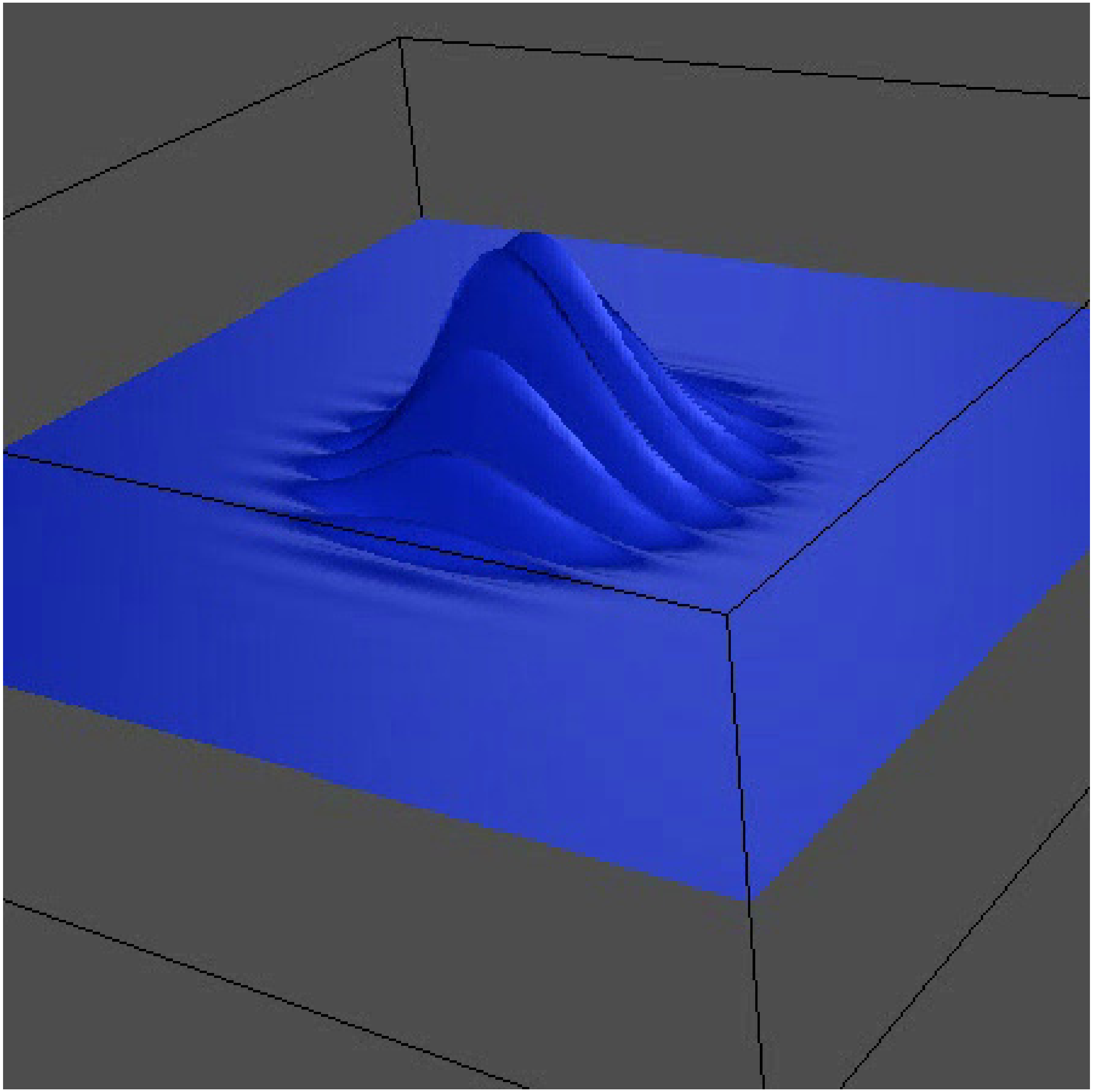
- The quantity Δ*x* is the length or spatial extent of a wave packet.
- Δp_x is a small range of momenta corresponding to the small range of frequencies within the wave packet.
- Any matter wave must obey the condition

$$\Delta x \Delta p_x \geq \frac{h}{2} \qquad \text{(Heisenberg uncertainty principle)}$$

This statement about the relationship between the position and momentum of a particle was proposed by Heisenberg in 1926. Physicists often just call it the **uncertainty principle.**

FIGURE 40.17 A snapshot graph of a wave packet.





Heisenberg Uncertainty principle:

Wave Packet:

$\Delta f \Delta t \sim 1 \Rightarrow \Delta \omega \Delta t \sim 2\pi$

A Traveling wave spread out in time + frequency is equivalent to being spread out in space (Δx) + spacial frequencies (Δk):

$\Delta x \Delta k \sim 2\pi$

+ $\Delta k = \frac{2\pi}{\Delta \lambda}$; $\frac{1}{\Delta \lambda} = \frac{\Delta p}{h}$

$\Rightarrow \Delta x \frac{\Delta p}{\hbar} \sim 1$

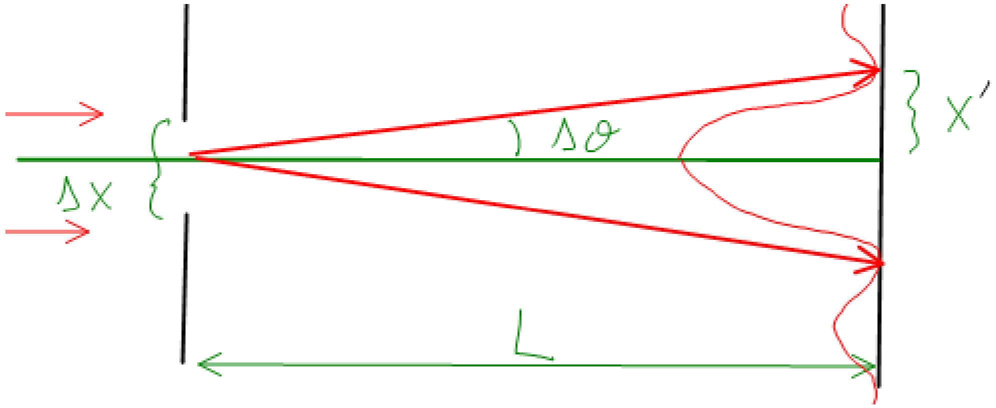
Modern gives $\Delta x \Delta p \geq \frac{\hbar}{2}$

Book gives $\Delta x \Delta p \geq \frac{h}{2}$

or $\Delta x \Delta p \sim h$

The Heisenberg Uncertainty Principle

- If we want to know *where* a particle is located, we measure its position x with uncertainty Δx.
- If we want to know *how fast* the particle is going, we need to measure its velocity v_x or, equivalently, its momentum p_x. This measurement also has some uncertainty Δp_x.
- You *cannot* measure both x and p_x simultaneously with arbitrarily good precision.
- Any measurements you make are limited by the condition that $\Delta x \Delta p_x \geq h/2$.
- **Our knowledge about a particle is *inherently* uncertain.**



$$\frac{\Delta x}{2m}\sin\Delta\theta = \frac{\lambda}{2}$$ Single Slit, first minimum $m=1$

$$\Rightarrow \sin\Delta\theta = \frac{\lambda}{\Delta x} \approx \tan\Delta\theta = \frac{x'}{L}$$ ①

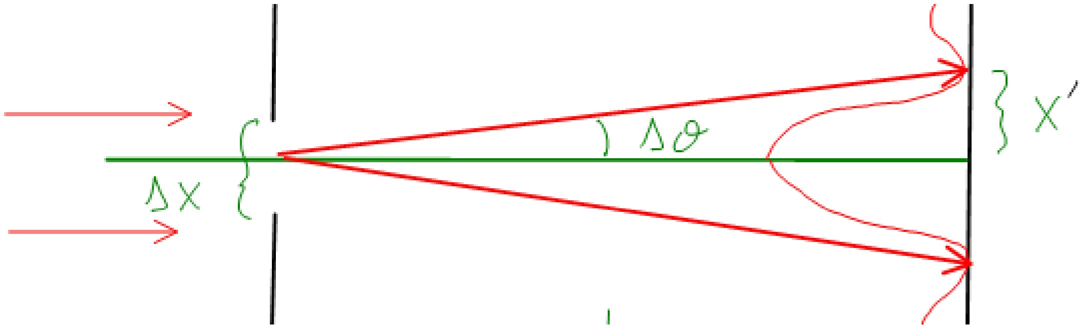
A particle that is "deflected" acquires vertical momentum.
For a particle to land at x requires:

$x' = V_x t$, t = time of flight from slit to screen

$$\approx \frac{L}{V_0} = \frac{L}{P_0/m}$$

$$V_x = \frac{P_x}{m}$$

$$\Rightarrow x' = \frac{mL}{P_0}\frac{P_x}{m} = \frac{L}{P_0}P_x \quad ; \quad P_0 = \frac{h}{\lambda}$$



$$P_0 = \frac{h}{\lambda} \quad \& \quad x' = \frac{L}{P_0} P_x$$

$$\sin\Delta\theta = \frac{\lambda}{\Delta x} \simeq \tan\Delta\theta = \frac{x'}{L} \qquad ①$$

For a particle to land within Central Maximum:

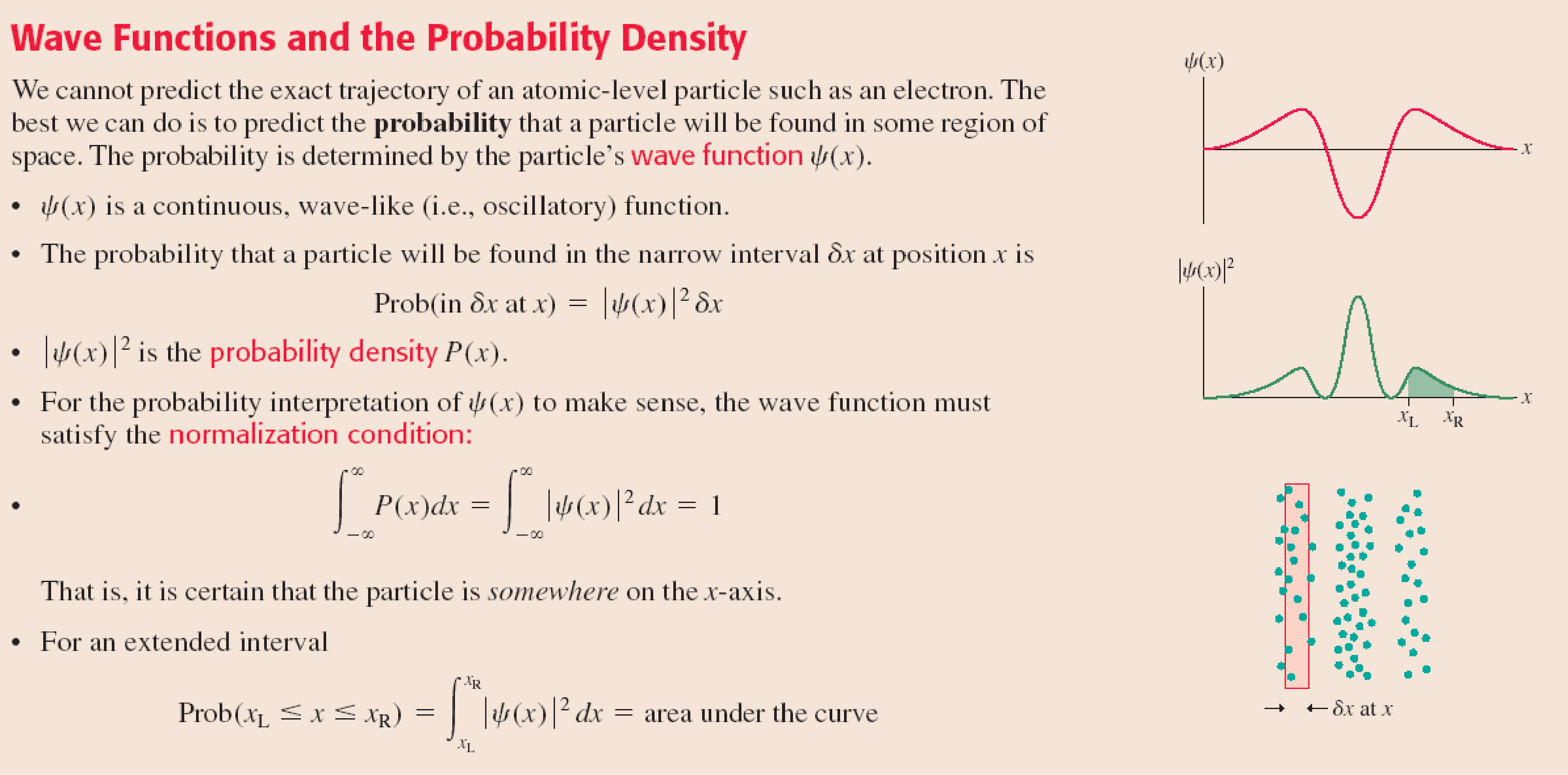
$$\Delta x' = \frac{L\lambda}{h} \Delta P_x$$

① $\Rightarrow$ $$\frac{\lambda}{\Delta x} = \frac{x'}{L} \quad \& \quad x' \le \Delta x'$$

$$\Rightarrow \quad \frac{\lambda}{\Delta x} \le \frac{\Delta x'}{L} \Rightarrow \Delta x \underbrace{\Delta x'}_{\frac{L\lambda}{h}\Delta P_x} \ge \lambda L$$

$$\Rightarrow \quad \Delta x \Delta P_x \ge h$$

General Principles



Wave Functions and the Probability Density

We cannot predict the exact trajectory of an atomic-level particle such as an electron. The best we can do is to predict the **probability** that a particle will be found in some region of space. The probability is determined by the particle's wave function $\psi(x)$.

- $\psi(x)$ is a continuous, wave-like (i.e., oscillatory) function.
- The probability that a particle will be found in the narrow interval δx at position x is

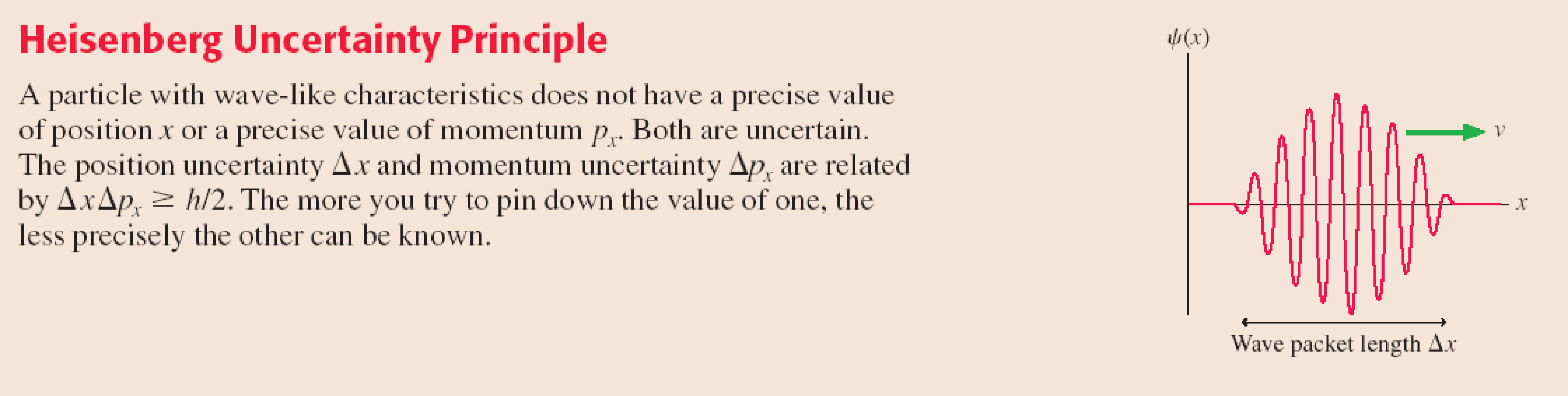
$$\text{Prob(in } \delta x \text{ at } x) = |\psi(x)|^2\,\delta x$$

- $|\psi(x)|^2$ is the probability density $P(x)$.
- For the probability interpretation of $\psi(x)$ to make sense, the wave function must satisfy the normalization condition:
- $$\int_{-\infty}^{\infty} P(x)dx = \int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1$$

  That is, it is certain that the particle is *somewhere* on the x-axis.
- For an extended interval

$$\text{Prob}(x_L \le x \le x_R) = \int_{x_L}^{x_R} |\psi(x)|^2\,dx = \text{area under the curve}$$

General Principles



Heisenberg Uncertainty Principle

A particle with wave-like characteristics does not have a precise value of position x or a precise value of momentum p_x. Both are uncertain. The position uncertainty Δx and momentum uncertainty Δp_x are related by $\Delta x \Delta p_x \geq h/2$. The more you try to pin down the value of one, the less precisely the other can be known.

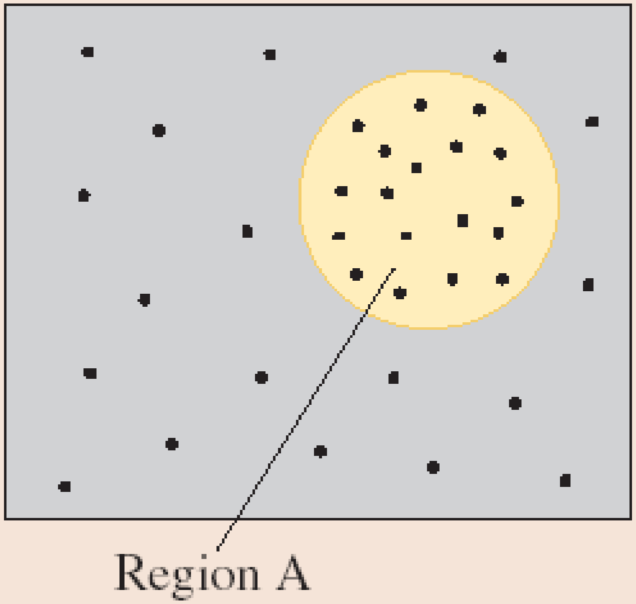
Wave packet length Δx

Important Concepts

The probability that a particle is found in region A is

$$P_A = \lim_{N_{tot} \to \infty} \frac{N_A}{N_{tot}}$$

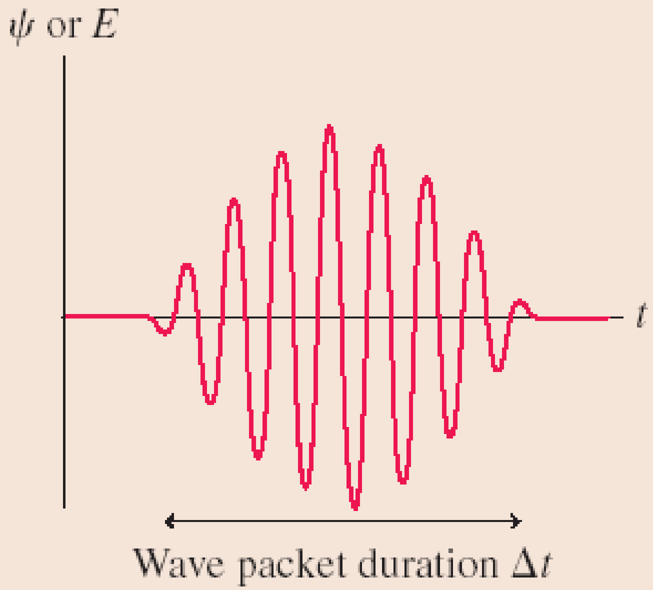
If the probability is known, the expected number of A outcomes in N trials is $N_A = NP_A$.



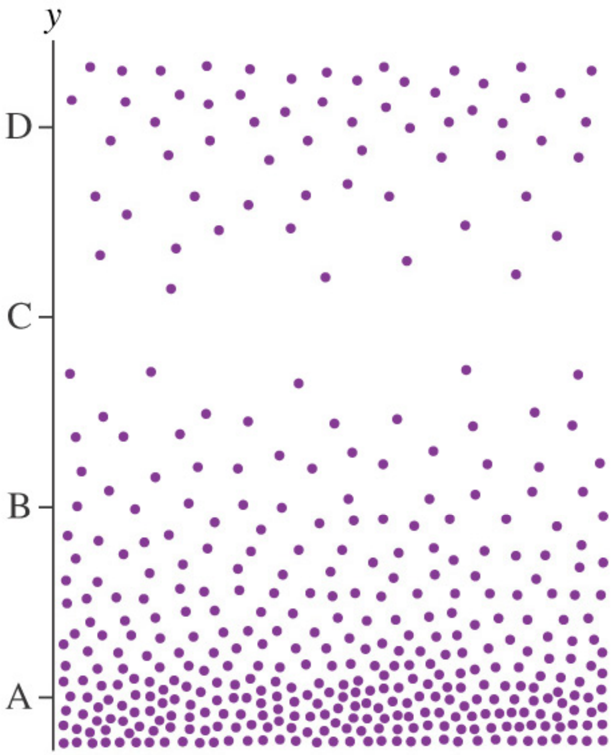
Important Concepts

A wave packet of duration Δt can be created by the superposition of many waves spanning the frequency range Δf. These are related by

$$\Delta f \Delta t \approx 1$$

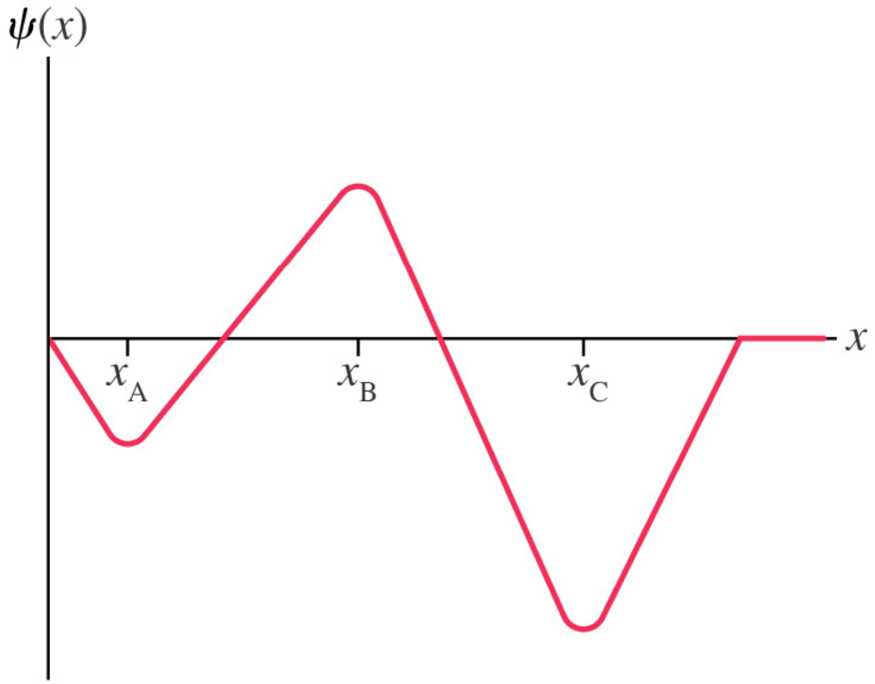


The figure shows the detection of photons in an optical experiment. Rank in order, from largest to smallest, the square of the amplitude function of the electromagnetic wave at positions A, B, C, and D.



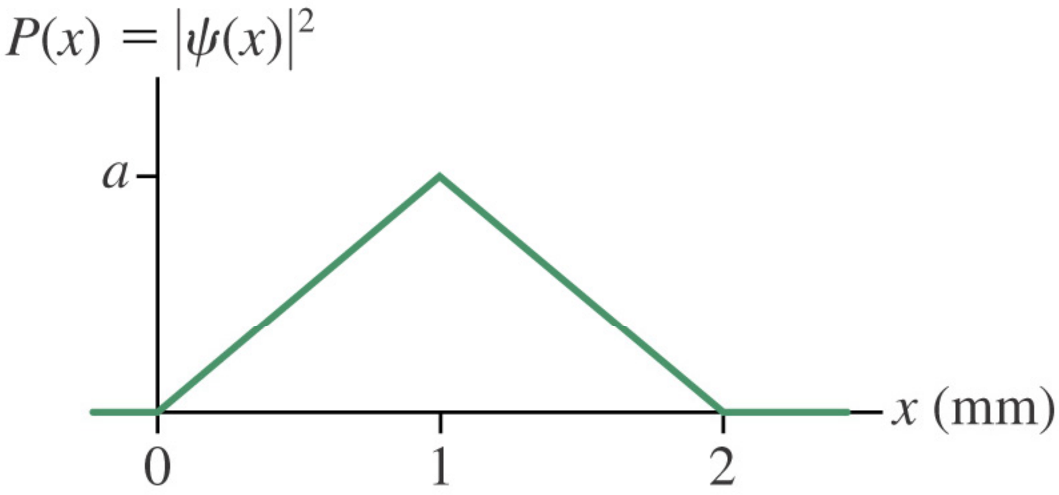
A. D > C > B > A
B. A > B > C > D
C. A > B = D > C
D. C > B = D > A

This is the wave function of a neutron. At what value of *x* is the neutron most likely to be found?



A. $x = 0$
B. $x = x_A$
C. $x = x_B$
D. $x = x_C$

The value of the constant *a* is



A. $a = 0.5\ \text{mm}^{-1/2}$.
B. $a = 1.0\ \text{mm}^{-1/2}$.
C. $a = 2.0\ \text{mm}^{-1/2}$.
D. $a = 1.0\ \text{mm}^{-1}$.
E. $a = 2.0\ \text{mm}^{-1}$.

What minimum bandwidth must a medium have to transmit a 100-ns-long pulse?

A. 100 MHz
B. 0.1 MHz
C. 1 MHz
D. 10 MHz
E. 1000 MHz