Chapter 37. Relativity

Topics:

- Relativity: What’s It All About?
- Galilean Relativity
- Einstein’s Principle of Relativity
- Events and Measurements
- The Relativity of Simultaneity
- Time Dilation
- Length Contraction
- The Lorentz Transformations
- Relativistic Momentum
- Relativistic Energy

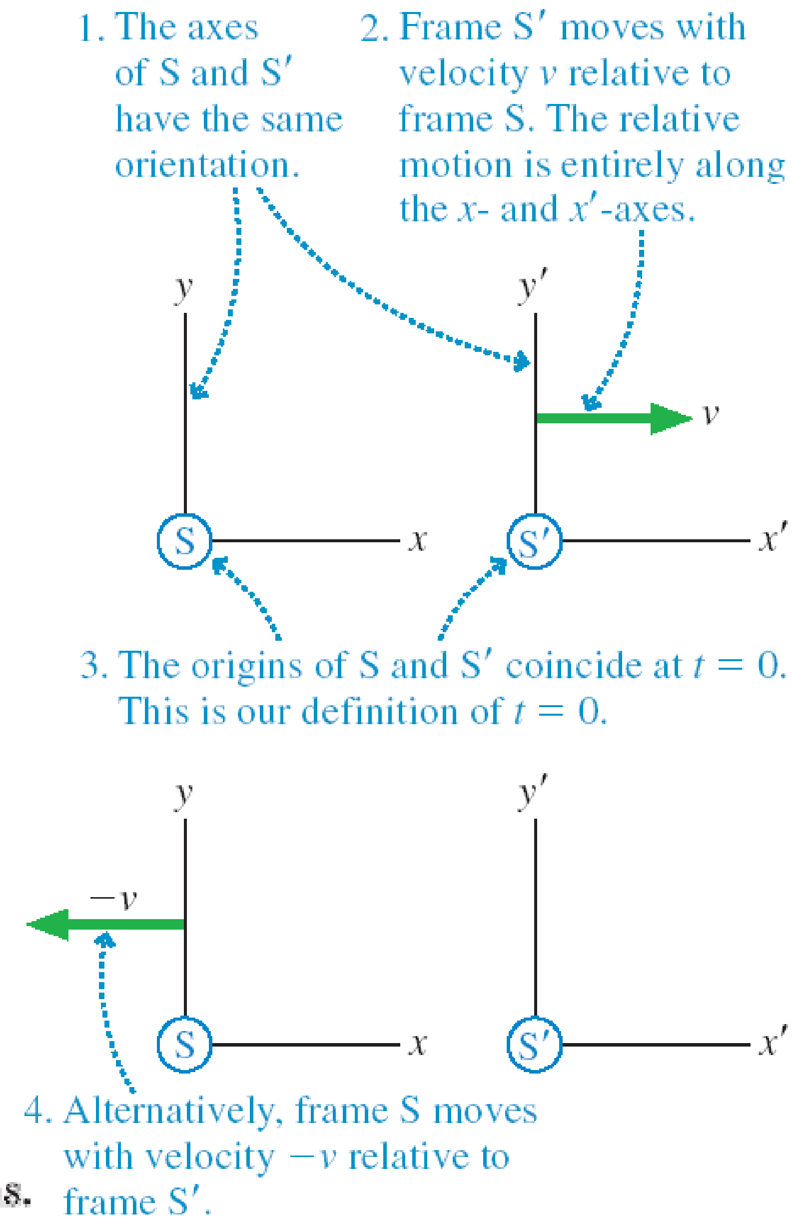
Reference frames

Defn: Inertial reference frame is one in body moves at constant velocity if the there is no force acting upon it

Two inertial reference frames, one traveling at constant velocity with respect to the other.

"special relativity" --- frames do not accelerate w.r.t. each other.

All experimenters that are at rest with one another share the same reference frame.



- A reference frame extends infinitely far in all directions.
- The experimenters are at rest in the reference frame.
- The number of experimenters and the quality of their equipment are sufficient to measure positions and velocities to any level of accuracy needed.

The Galilean Transformations

Chapter 4:

S → *x*, *y*, *z*

S′ → *x*′, *y*′, *z*′.

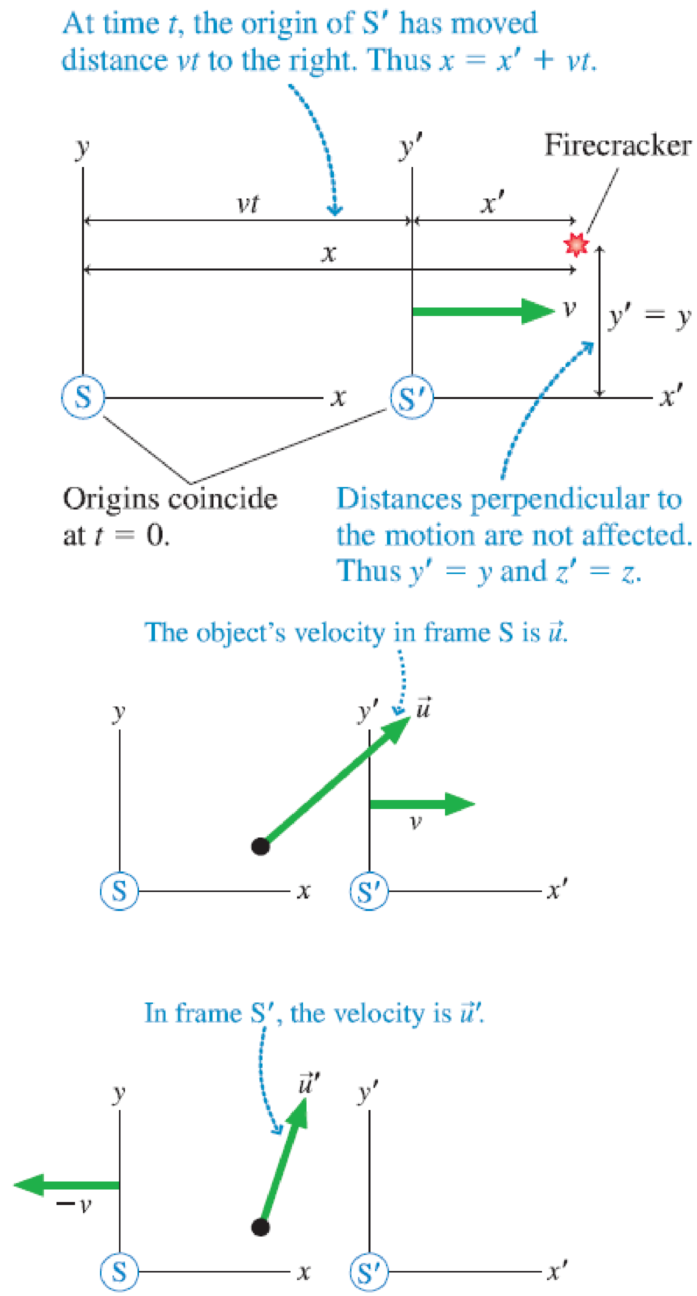
S′ moves with velocity *v* relative to S along the *x*-axis.

The *Galilean transformations of position* are:

$$x = x' + vt \qquad\qquad x' = x - vt$$
$$y = y' \qquad \text{or} \qquad y' = y$$
$$z = z' \qquad\qquad z' = z$$

The *Galilean transformations of velocity* are:

$$u_x = u_x' + v \qquad\qquad u_x' = u_x - v$$
$$u_y = u_y' \qquad \text{or} \qquad u_y' = u_y$$
$$u_z = u_z' \qquad\qquad u_z' = u_z$$



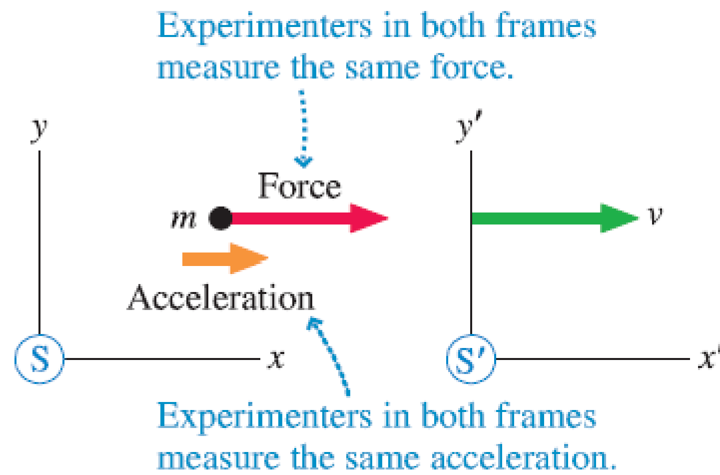
The Galilean Transformations

Chapter 4:

$$\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \end{aligned} \qquad \text{or} \qquad \begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

$$\begin{aligned} u_x &= u'_x + v \\ u_y &= u'_y \\ u_z &= u'_z \end{aligned} \qquad \text{or} \qquad \begin{aligned} u'_x &= u_x - v \\ u'_y &= u_y \\ u'_z &= u_z \end{aligned}$$

FIGURE 37.5 Experimenters in both reference frames test Newton's second law by measuring the force on a particle and its acceleration.



$$a' = \frac{du'}{dt} = \frac{du}{dt} = a$$

$$F' = F$$

Galilean principle of relativity The laws of mechanics are the same in all inertial reference frames.

Using Galilean Transformations

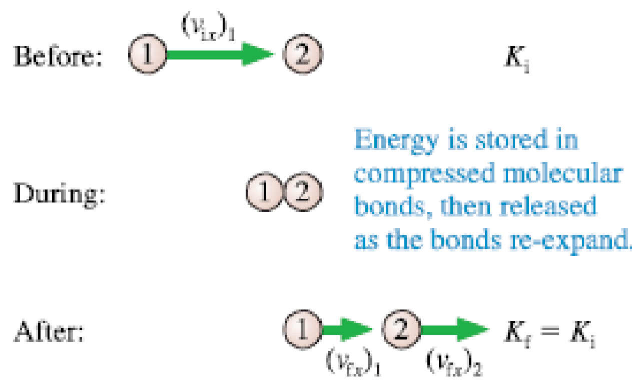
Chapter 10:

Elastic collisions and conservation of momentum

momentum conservation: $m_1(v_{fx})_1 + m_2(v_{fx})_2 = m_1(v_{ix})_1$

energy conservation: $\frac{1}{2}m_1(v_{fx})_1^2 + \frac{1}{2}m_2(v_{fx})_2^2 = \frac{1}{2}m_1(v_{ix})_1^2$

FIGURE 10.24 A perfectly elastic collision.



$$(v_{fx})_1 = \frac{m_1 - m_2}{m_1 + m_2}(v_{ix})_1$$
$$(v_{fx})_2 = \frac{2m_1}{m_1 + m_2}(v_{ix})_1$$
(perfectly elastic collision with ball 2 initially at rest)

Transform velocity to where one object is at rest w.r.t. other, solve collision problem, and then transform back

$$u'_{1f} = \frac{m_1 - m_2}{m_1 + m_2}u'_{1i}$$

$$u'_{2f} = \frac{2m_1}{m_1 + m_2}u'_{1i}$$

Einstein's Principle of Relativity

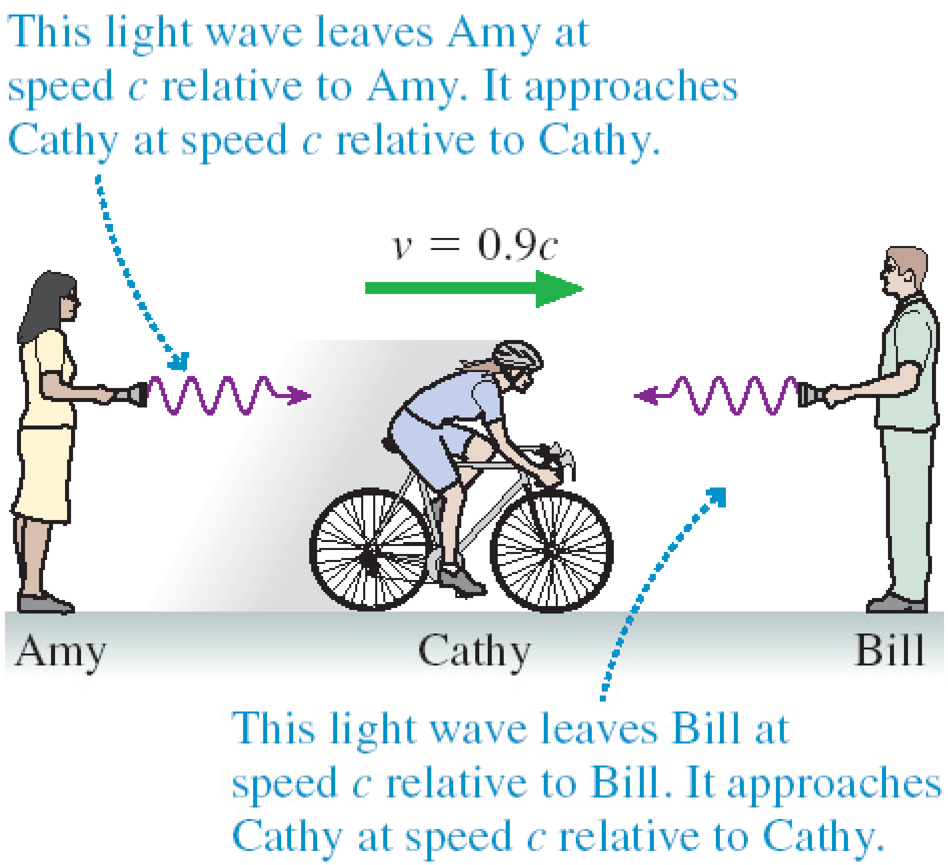
Principle of relativity All the laws of physics are the same in all inertial reference frames.

- Maxwell's equations are true in all inertial reference frames.
- Maxwell's equations predict that electromagnetic waves, including light, travel at speed $c = 3.00 \times 10^8$ m/s.
- Therefore, **light travels at speed *c* in all inertial reference frames.**

Phenomenological Proof:

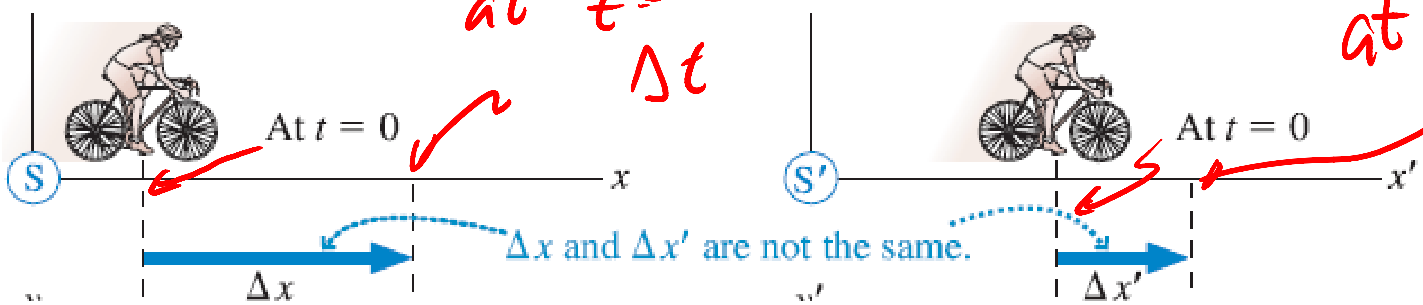
1. Thorndike and Kennedy: Interferometer (similar to Michelson-Morley Expt); No relative fringe shift when earth moves in orbit 6 months apart → $\Delta v \sim 60$ km/s

FIGURE 37.9 Light travels at speed c in all inertial reference frames, regardless of how the reference frames are moving with respect to the light source.



at $t = \Delta t$

at $t = \Delta t$



"Stationary" earth frame

driving in car (slower) than bike

$$\Delta x' < \Delta x \quad \therefore \quad u = \frac{\Delta x}{\Delta t} > u' = \frac{\Delta x'}{\Delta t}$$

If the Bike were a light wave $\Rightarrow u = u' = c$

$\Rightarrow \Delta t' < \Delta t$!! ($\Delta t'$ is "Δt" in frame S′)

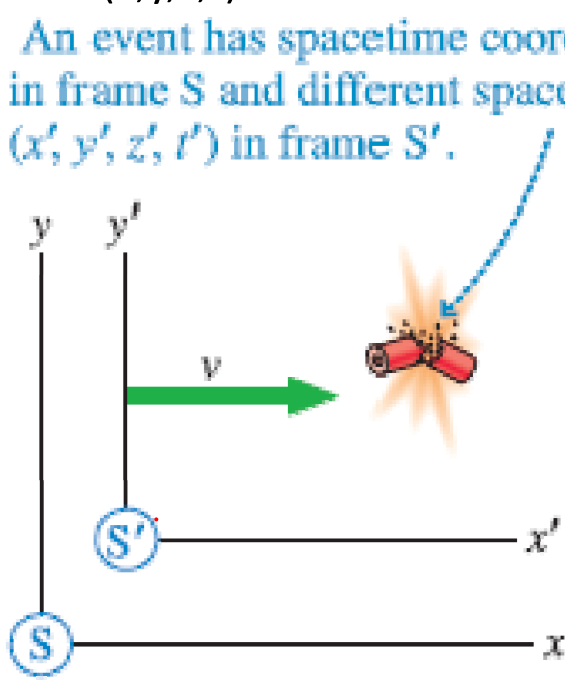
Relativity: Analysis of time

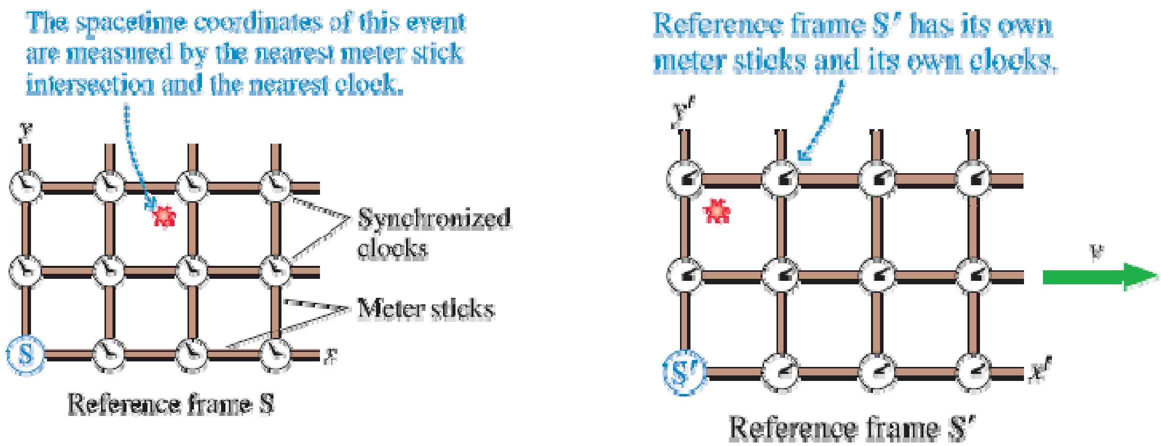
It was always assumed that time is the same in all reference frames

The implications of the speed of light being the same in all reference frames is that space and time MUST change in different reference frames.

To analyze the situation, we make some definition and set up our reference frames and clocks.

Event: Physical activity that takes place at a definite point in space and time
Space time coordinates: (x,y,z,t)





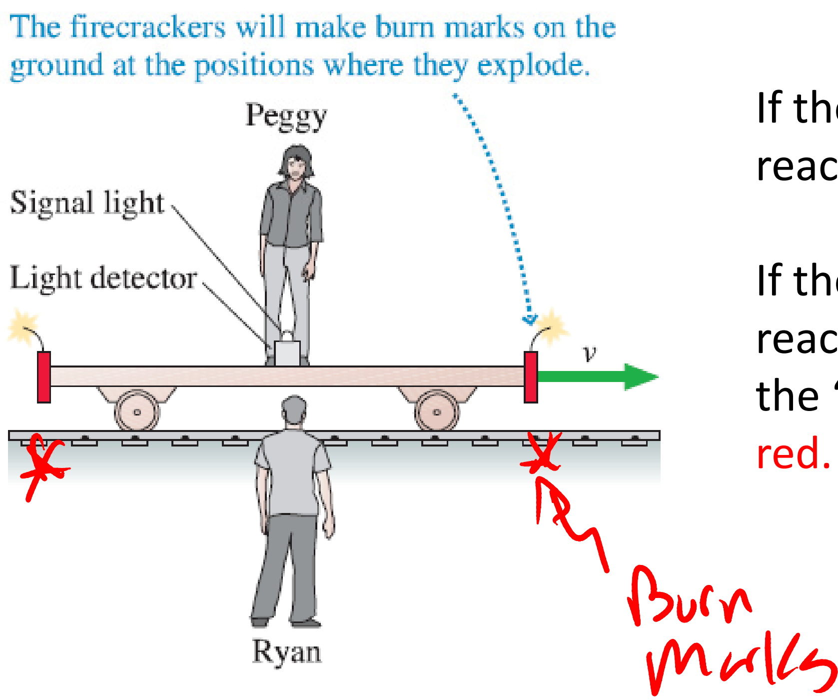
- The (x, y, z) coordinates of an event are determined by the intersection of the meter sticks closest to the event.
- The event's time t is the time displayed on the clock nearest the event.

1. The clocks and meter sticks in each reference frame are imaginary, so they have no difficulty passing through each other.
2. Measurements of position and time made in one reference frame must use only the clocks and meter sticks in that reference frame.
3. There's nothing special about the sticks being 1 m long and the clocks 1 m apart. The lattice spacing can be altered to achieve whatever level of measurement accuracy is desired.
4. We'll assume that the experimenters in each reference frame have assistants sitting beside every clock to record the position and time of nearby events.
5. Perhaps most important, t is the time at which the event *actually happens*, not the time at which an experimenter sees the event or at which information about the event reaches an experimenter.
6. All experimenters in one reference frame agree on the spacetime coordinates of an event. In other words, **an event has a unique set of spacetime coordinates in each reference frame.**

Relativity: Analysis of time

Two events are **simultaneous** if they occur at two different places but at the same time as measured in the ***SAME*** reference frame

FIGURE 37.16 A railroad car traveling to the right with velocity v.



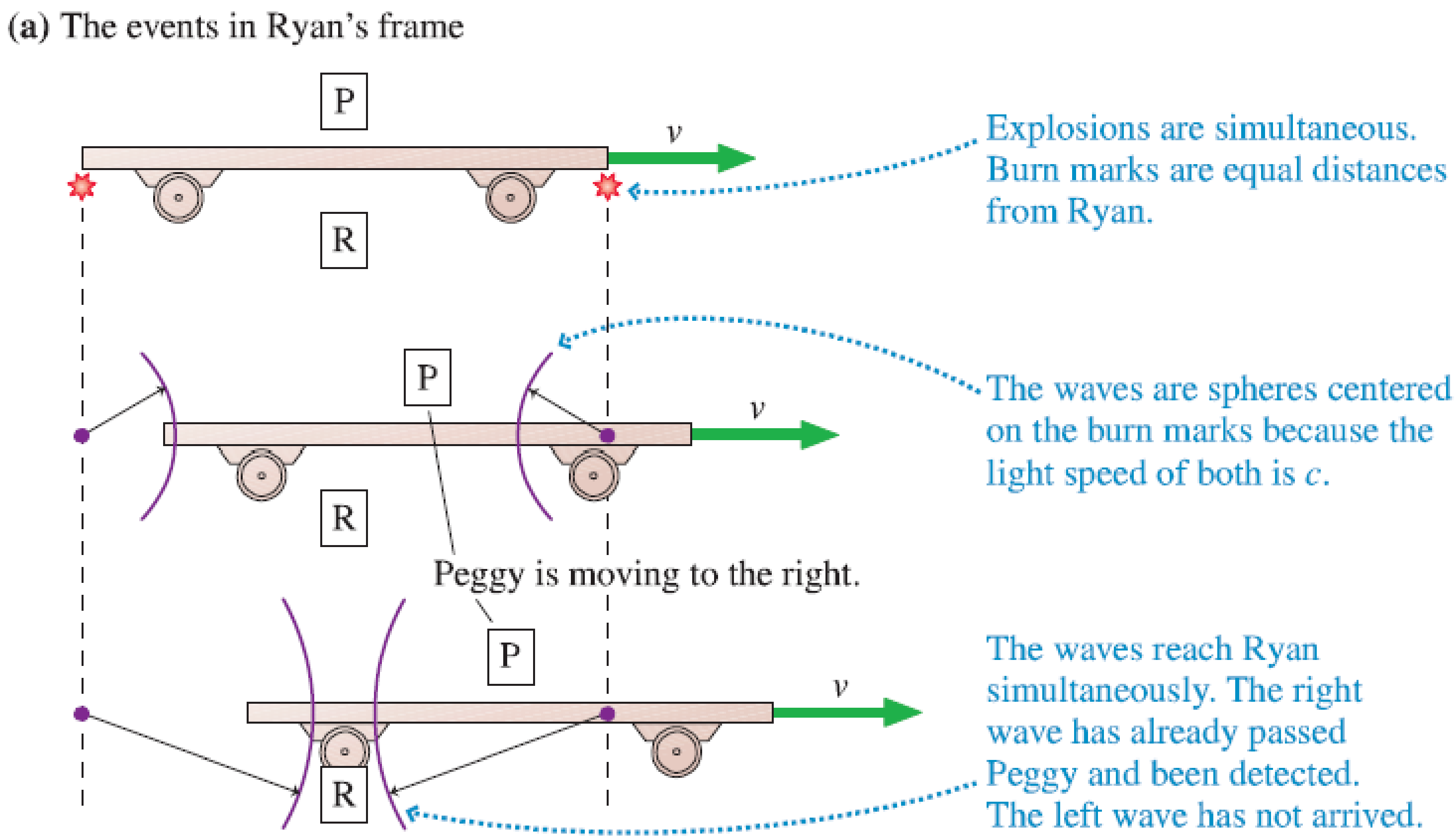
If the light from the “right” (front) firecracker reaches detector first, the light is green.

If the light from the “left” (back) firecracker reaches the detector first or at the same time as the “right” (front) firecracker, then the light turns red.

1. Ryan detected the flashes simultaneously.
2. Ryan was halfway between the firecrackers when they exploded.
3. The light from the two explosions traveled toward Ryan at equal speeds.

The conclusion that the explosions were simultaneous in Ryan’s reference frame is unassailable. The light is green.

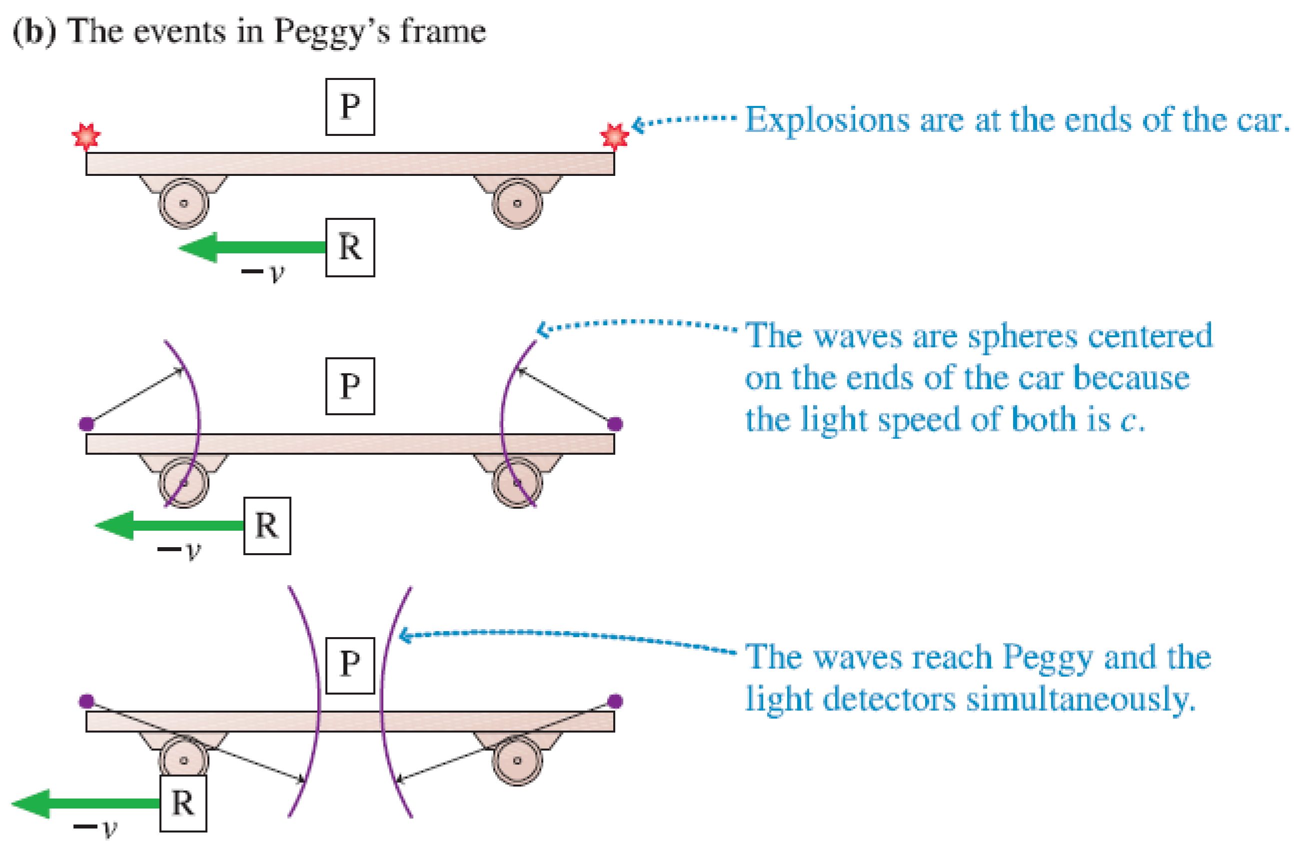
Ryan’s Reference frame analysis



Light is Green since “right” firecracker light reaches detector first.

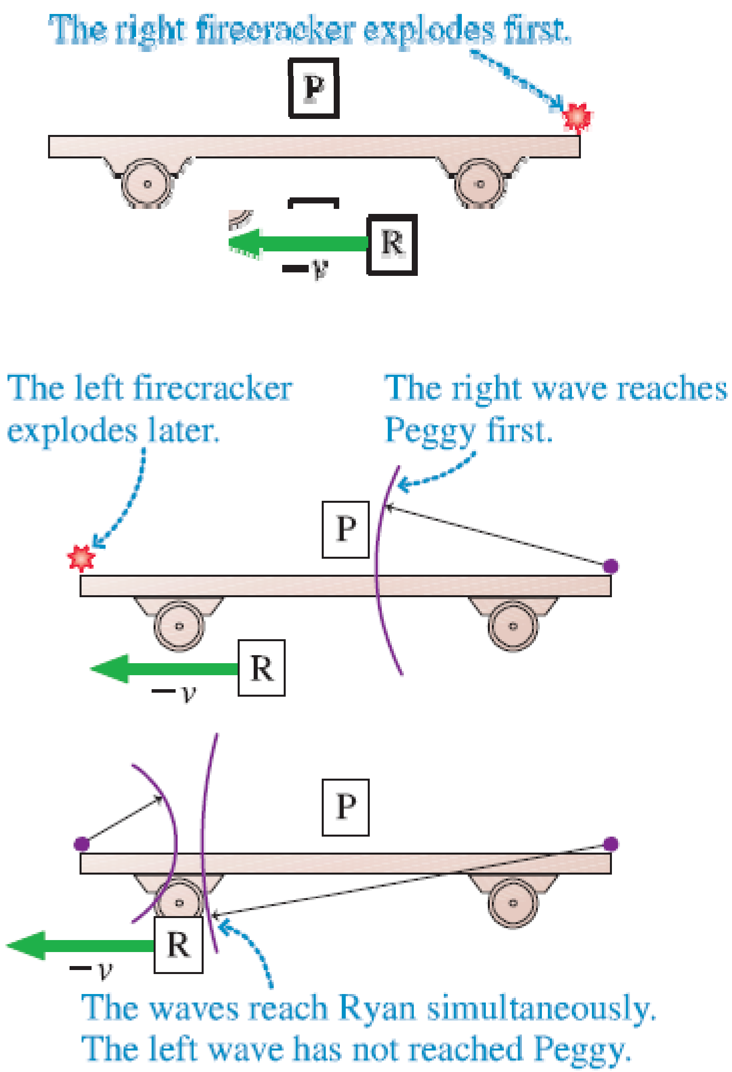
Peggy's Reference frame WRONG analysis

IF you ASSUME that the firecrackers are simultaneous in Peggy's frame as well, a false assumption, then the light should be RED



Light should be RED since light reaches detector at same time, which is wrong.

Peggy's Reference frame Correct analysis



Ryan ***must*** detect the two waves simultaneously. Everything flows from this idea.

Since the wave from the right firecracker must travel further to reach Ryan IN PEGGY's FRAME, it must have exploded before the left firecracker IN PEGGY's FRAME.

The firecrackers are NOT simultaneous in Peggy's frame, although they are in Ryan's frame

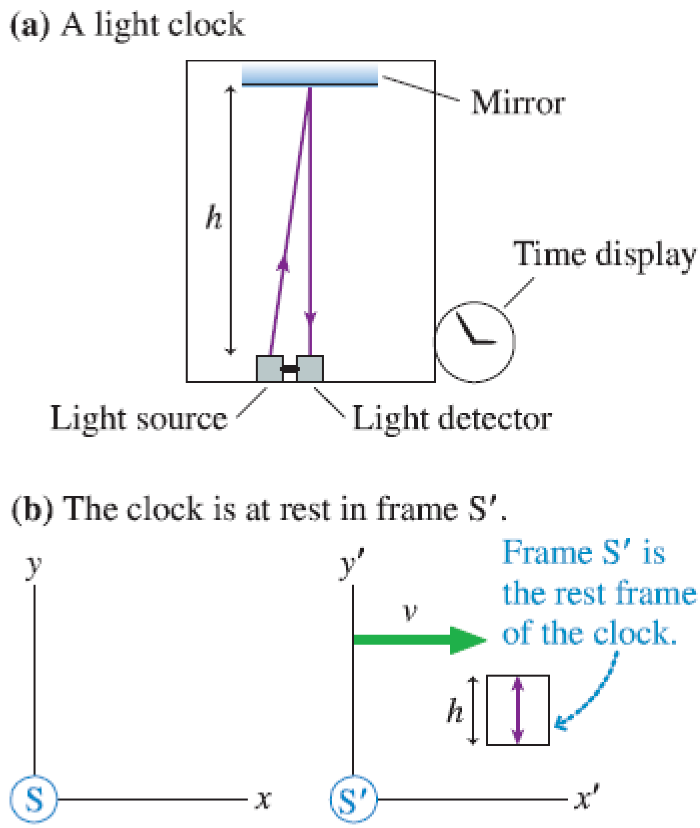
The light is green.

"simultaneity" is relative --- that is, whether two events occur at the same time is dependent upon your reference frame

The two firecrackers *really* explode at the same instant of time in Ryan's reference frame. And the right firecracker *really* explodes first in Peggy's reference frame. It's not a matter of when they see the flashes. Our conclusion refers to the times at which the explosions actually occur.

Analysis of time – Time dilation

FIGURE 37.19 The ticking of a light clock can be measured by experimenters in two different reference frames.



"Light" clock – Light pulse fires from source, bounces off of mirror and back into detector, and immediately fires off the next pulse.

Assume the source and detector are at the same place.

Our conclusions based on our analysis of the "light clock" will be true for any type of clock (heart beat, grandfather clock, digital watch, etc.)

Goal: Compare the differences between two time intervals of 1 tick of the clock in frame S and S'

We already know that the time intervals are different since Ryan measured $\Delta t=0$ between two events (simultaneous) while Peggy measured nonzero

Event 1: Emission of light pulse
Event 2: Detection of light pulse

(b) The clock is at rest in frame S′.

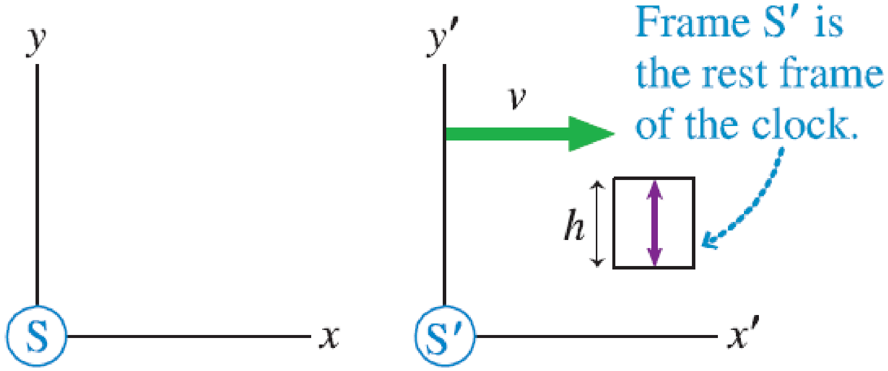
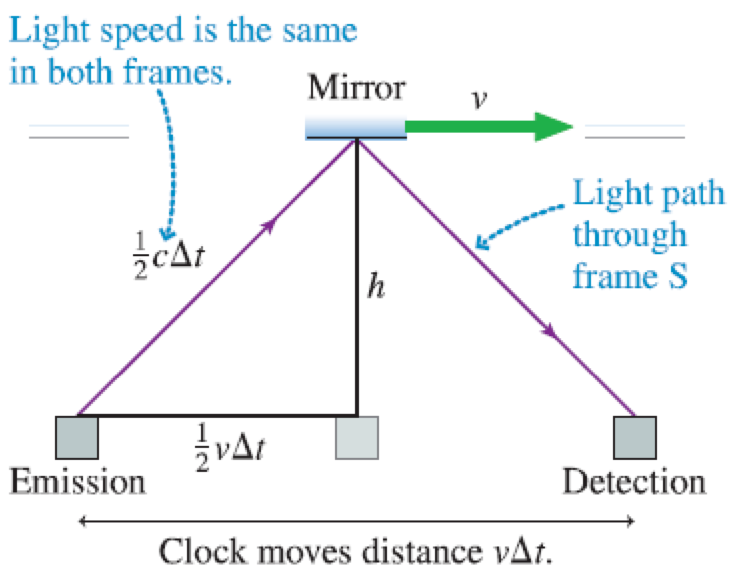


FIGURE 37.21 A light clock analysis in which the speed of light is the same in all reference frames.



frame S′

$$c = \frac{\Delta l'}{\Delta t'} \Rightarrow \Delta t' = \frac{2h}{c}$$

frame S:

$$c = \frac{\Delta l}{\Delta t} \; ; \; \Delta l = 2\sqrt{h^2 + \left(\tfrac{1}{2} v \Delta t\right)^2}$$

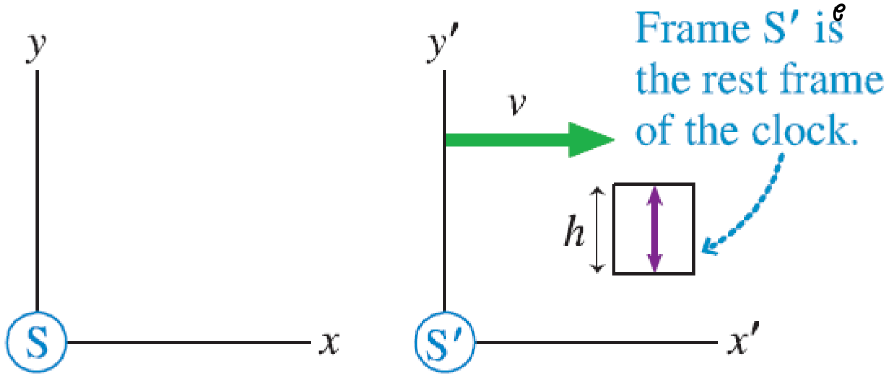
$$\Rightarrow \Delta t = \frac{\Delta l}{c} \Rightarrow \Delta t^2 = \frac{1}{c^2}\left[(2h)^2 + (v\Delta t)^2\right]$$

$$\Rightarrow (\Delta t)^2 \left(1 - \left(\frac{v}{c}\right)^2\right) = \left(2h/c\right)^2$$

$$\Rightarrow \Delta t = (1-\beta^2)^{-1/2}\left(2h/c\right)$$

$$\therefore \Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}} \; , \; \beta \equiv \frac{v}{c}$$

(b) The clock is at rest in frame S'.

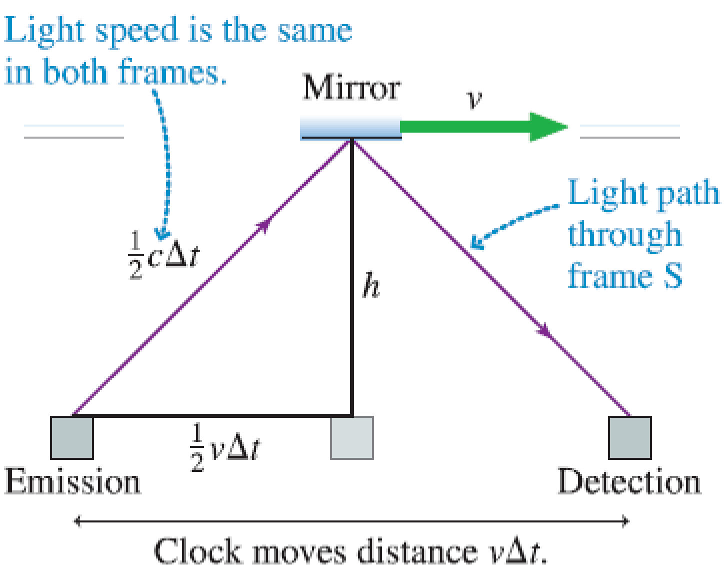


$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}}, \quad \beta \equiv \frac{v}{c}$$

Shortest time between ticks is in the frame where the clock is at rest --- That is, the frame in which the two events (emission and detection) are measured with the ***same*** clock. In this case, this is called the proper time and is notated as $\Delta\tau$.

MORE time passes per tick in frame S in which the clock is moving than in the stationary frame S'.

FIGURE 37.21 A light clock analysis in which the speed of light is the same in all reference frames.



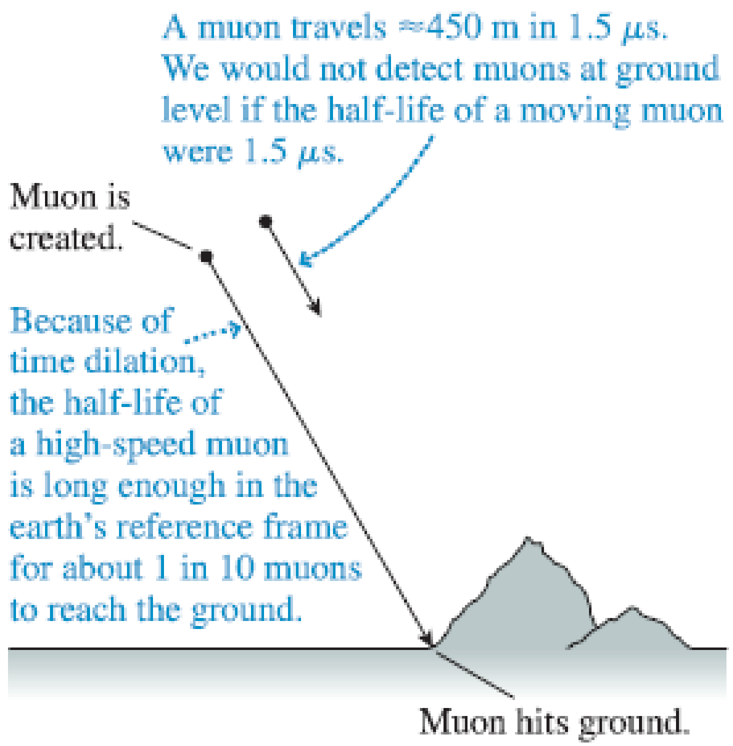
$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\beta^2}} \geq \Delta\tau \qquad \text{(time dilation)}$$

"time dilation"

Time dilation Evidence

1. 1971 --- Clock on ground synchronized with clock on plane; Plane takes off and travels around the world back to original location. The clock on the plane was slower (by 200ns)

2. Muon radioactive decay --- Stationary Muons decays with half life 1.5 us in atmosphere. Fast moving muons from cosmic rays should not make it to the ground. A large percentage make it due to time dilation



Twin Paradox

Two twins, call them Earl and Roger:
Earl is on Earth
Roger is in Rocket

Roger takes off at relativistic velocity to Jupiter and back. Both Roger and Earl measure the take off event and the return event with the ***same*** clock in their respective reference frames.

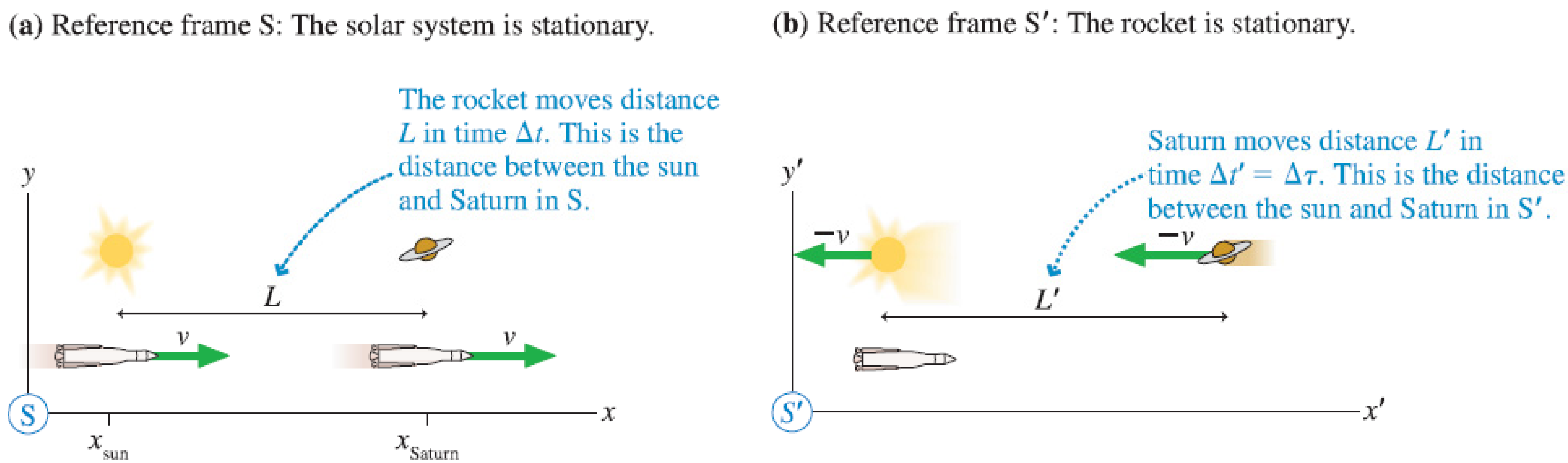
Who is it that is measuring the proper time? Both Roger and Earl think they are measuring Proper time and think that the other guy should be younger (slower clock) than themselves.

There is another intermediary event: the rocket decelerates and accelerates to turn around and go back to earth. Since this event is not measured by Earl with the same clock, Earl is not measuring proper time. Roger measures proper time. Therefore Roger is younger than Earl upon his return.

Caveat: The ‘lost’ time must be associated with the acceleration and deceleration….

Length Contraction

FIGURE 37.25 L and L' are the distances between the sun and Saturn in frames S and S′.



Rocket (S' frame) measures proper time: Both events with one clock (same place)

$$v = \frac{L}{\Delta t} = \frac{L'}{\Delta t'} \Rightarrow \frac{L}{\Delta t} = \frac{L'}{\Delta \tau} = \frac{L'}{\sqrt{1-\beta^2}\Delta t}$$

$$L' = \sqrt{1-\beta^2}L$$

"Length Contraction"

Proper length – length measured in frame where object is at rest

$$L' = \sqrt{1-\beta^2}\,\ell \le \ell$$

Length Contraction

$$L' = \sqrt{1 - \beta^2}L$$

The conclusion that space is different in reference frames moving relative to each other is a direct consequence of the fact that time is different. Experimenters in both reference frames agree on the relative velocity v, leading to Equation 37.12: $v = L/\Delta t = L'/\Delta t'$. We had already learned that $\Delta t' < \Delta t$ because of time dilation. Thus L' *has* to be less than L. That is the only way experimenters in the two reference frames can reconcile their measurements.

8. As the meter stick in **FIGURE Q37.8** flies past you, you simultaneously measure the positions of both ends and determine that $L < 1$ m.

FIGURE Q37.8

a. To an experimenter in frame S′, the meter stick's frame, did you make your two measurements simultaneously? If not, which end did you measure first? Explain.

b. Can experimenters in frame S′ give an explanation for why your measurement is less than 1 m?

a) Xb,

a) This is Like Peg + Ryan on the train.

If Ryan observes firecrackers are simultaneous

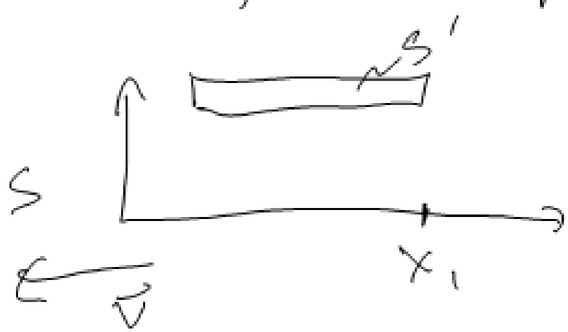
⇒ light from "Right" firecracker reaches Peg first

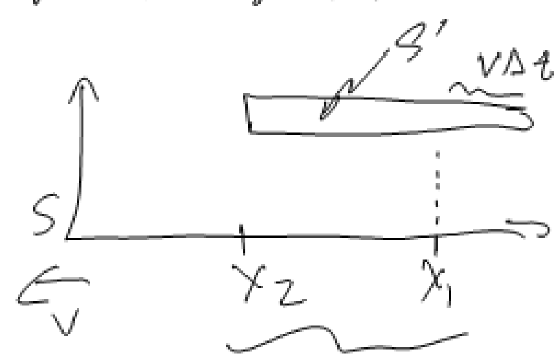
⇒ Peggy observes "Right" firecracker goes off first

If Ryan measures "Right" + "Left" of meter stick simultaneously,

⇒ in moving meter stick frame the "Right" end was measured first + the "left" end was measured later

b) ∴





$x_2 - x_1 < L$

9. A 100-m-long train is heading for an 80-m-long tunnel. If the train moves sufficiently fast, is it possible, according to experimenters on the ground, for the entire train to be inside the tunnel at one instant of time? Explain.

Train is $L_0 =$ 100 m long in train's rest frame,

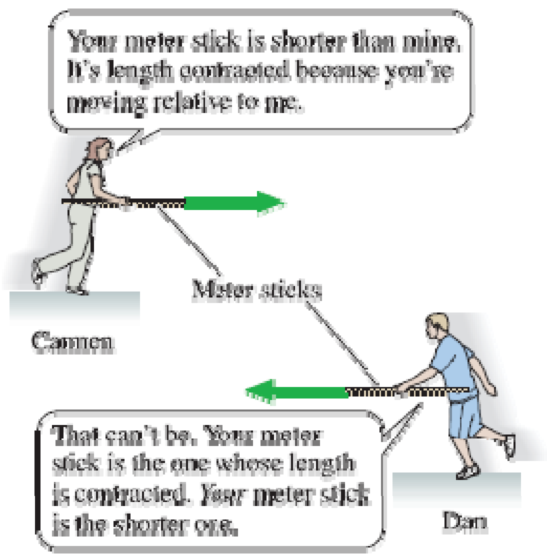
From tunnel frame, $L < L_0$, $L = L_0\sqrt{1-\beta^2}$

Like problem #8: Consider firecrackers go off at front & back of train simultaneously in tunnel frame $\Rightarrow$ measure time delay between light pulses $\rightarrow$ length of train

In train's frame, front firecracker goes off first

Length Contraction

FIGURE 37.26 Carmen and Dan each measure the length of the other's meter stick as they move relative to each other.



Carmen's Reference frame:

event 1: Front of Dan's meter stick at front of Carmen's meter stick

event 2: Back of Dan's meter stick @ at front of Carmen's meter stick

Dan's Reference frame

event 1: Front of Carmen's meter stick at front of Dan's meter stick

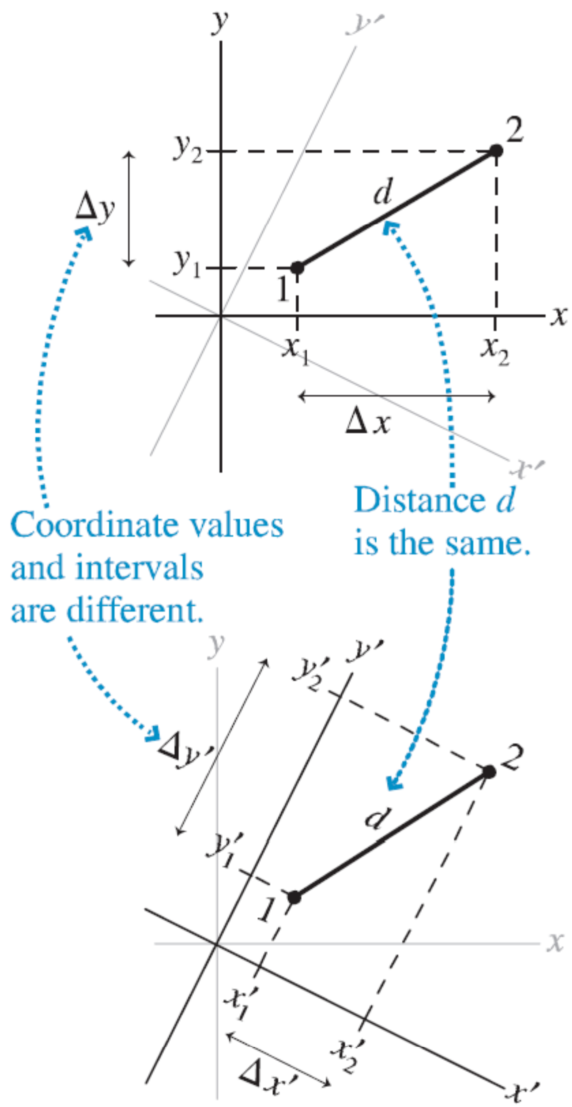
event 2: Back of Carmen's meter stick @ at front of Dan's meter stick

Not the same Events

Both observe Length Contraction

Space-time interval

Measurements in the xy-system



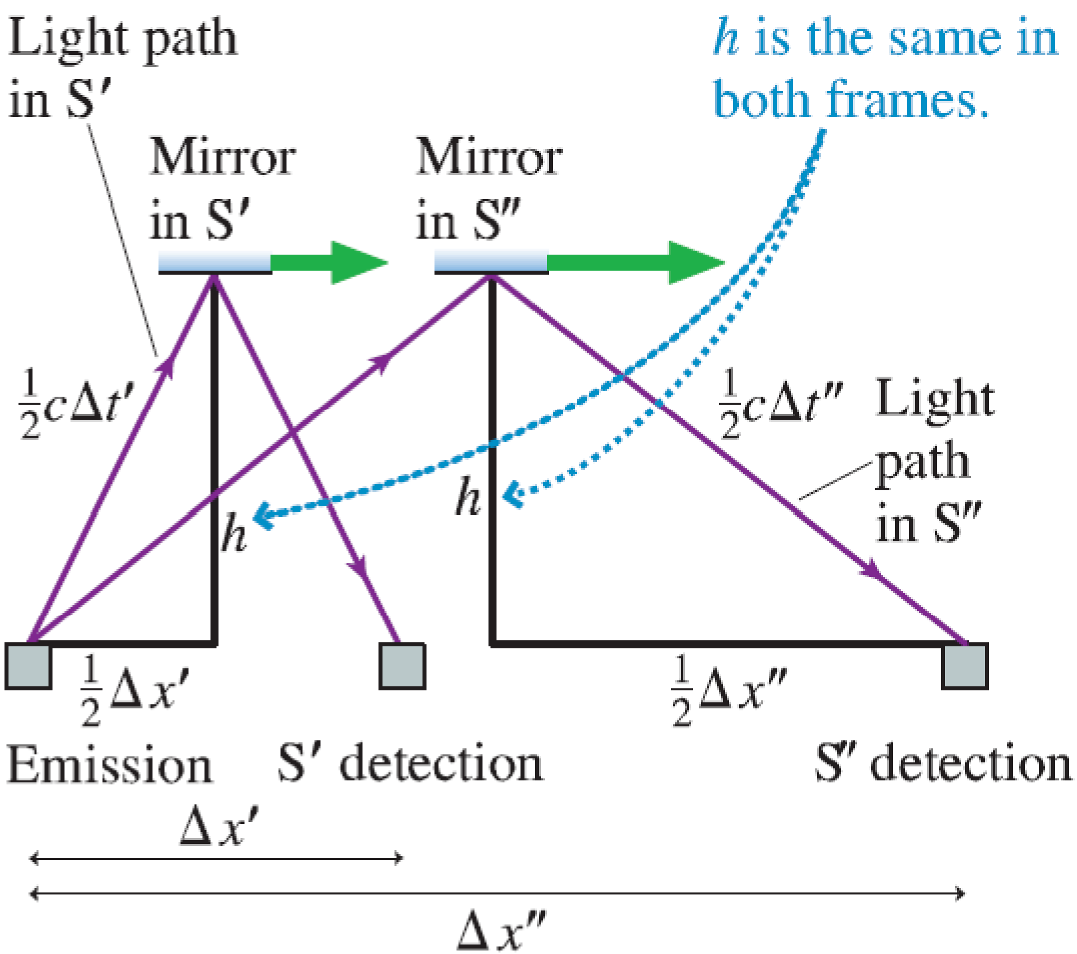
Measurements in the $x'y'$-system

$$d^2 = (\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + (\Delta y')^2$$

Invariant with respect to translation and rotation – meaning it has the same value no matter how your reference frame is rotated or translated

Space-time interval

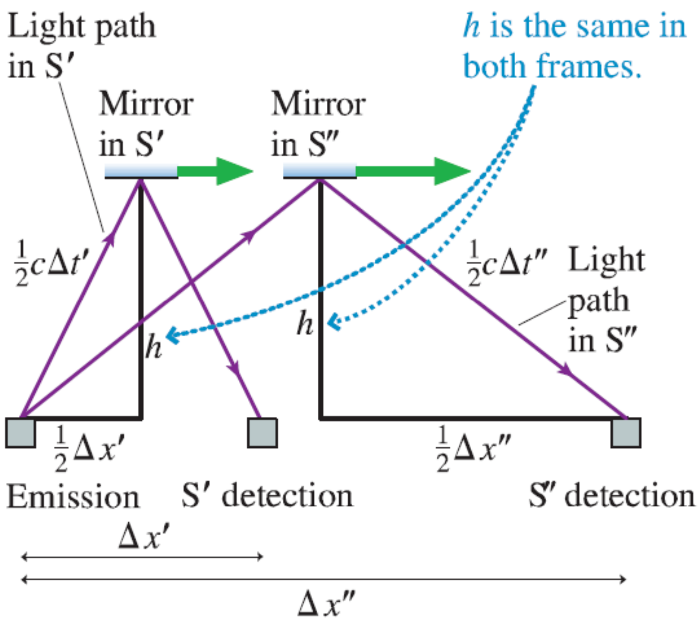
FIGURE 37.28 The light clock seen by experimenters in reference frames S′ and S″.



h is invariant no matter how fast the reference frame is moving

Space-time interval

FIGURE 37.28 The light clock seen by experimenters in reference frames S′ and S″.



h is invariant no matter how fast the reference frame is moving

$$h^2 = \left(\frac{1}{2}c\,\Delta t'\right)^2 - \left(\frac{1}{2}\Delta x'\right)^2 = \left(\frac{1}{2}c\,\Delta t''\right)^2 - \left(\frac{1}{2}\Delta x''\right)^2$$

$$c^2(\Delta t')^2 - (\Delta x')^2 = c^2(\Delta t'')^2 - (\Delta x'')^2$$

spacetime interval s

$$s^2 = c^2(\Delta t)^2 - (\Delta x)^2$$

S is an invariant in relativity --- all observers will measure the same spacetime interval between two events

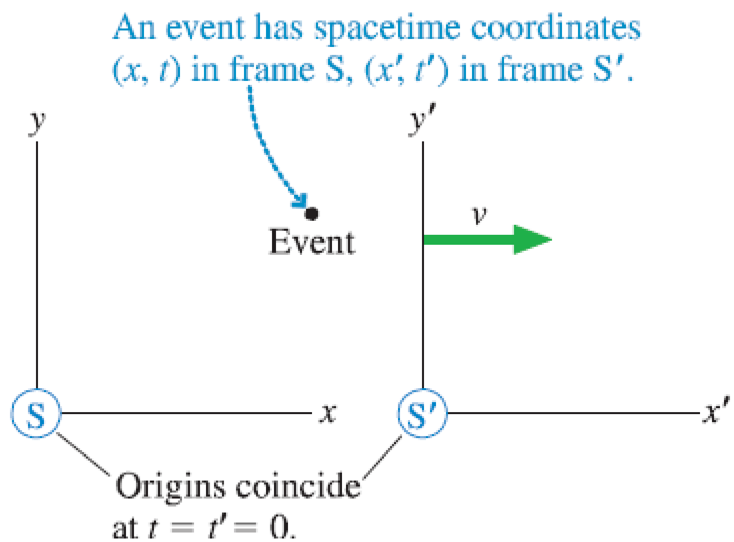
EXAMPLE 37.7 Using the spacetime interval

A firecracker explodes at the origin of an inertial reference frame. Then, 2.0 μs later, a second firecracker explodes 300 m away. Astronauts in a passing rocket measure the distance between the explosions to be 200 m. According to the astronauts, how much time elapses between the two explosions?

Lorentz transformation

We'll continue to use reference frames in the standard orientation of FIGURE 37.29. The motion is parallel to the x- and x'-axes, and we *define* $t = 0$ and $t' = 0$ as the instant when the origins of S and S′ coincide.

iff $x = x' = 0$ → like a "first" event

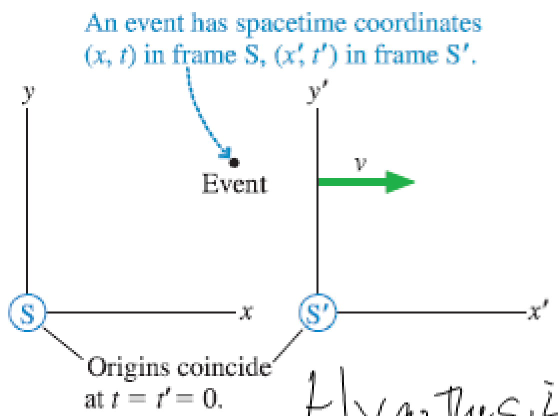


Given $x = x' = 0$ @ $t = t' = 0$, How do we find spacetime coordinate of a (second) event?

Must Have

1. Agree with the Galilean transformations in the low-speed limit $v \ll c$.
2. Transform not only spatial coordinates but also time coordinates.
3. Ensure that the speed of light is the same in all reference frames.

Lorentz transformation



Event 1: A flash of light is emitted from the origin of both reference frames ($x = x' = 0$) at the instant they coincide ($t = t' = 0$).

Event 2: The light strikes a light detector. The spacetime coordinates of this event are (x, t) in frame S and (x', t') in frame S′.

Hypothesis : $x' = \gamma(x - vt)$ & $x = \gamma(x' + vt')$

Looks alot like Galilean transformation.

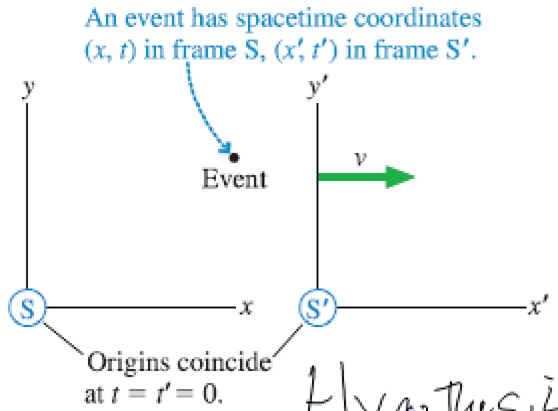
Note that as $v \to 0$, we must have $\gamma \to 1$ & $t \to t'$ to recover Galilean transformation

Speed of light is the same in both reference frames, & a light pulse goes off at the origin @ time zero in both frames!

$\Rightarrow$ $x' = ct'$ = How far light travels to detector in frame S′

$x = ct$ = How far light travels to detector in frame S

Lorentz transformation



Event 1: A flash of light is emitted from the origin of both reference frames ($x = x' = 0$) at the instant they coincide ($t = t' = 0$).

Event 2: The light strikes a light detector. The spacetime coordinates of this event are (x, t) in frame S and (x', t') in frame S′.

Hypothesis : $x' = \gamma(x - vt)$ & $x = \gamma(x' + vt')$

$\Rightarrow$ $x' = ct'$ = How far light travels to detector in frame S′

$x = ct$ = How far light travels to detector in frame S

Use Hypothesis to derive what γ has to be to satisfy "the speed of light is constant in all frames" condition.

Hypothesis: $x' = \gamma(x - vt)$ & $x = \gamma(x' + vt')$

$\Rightarrow$ $x' = ct'$ = How far light travels to detector in frame S'

$x = ct$ = How far light travels to detector in frame S

$$x' = ct' = \gamma(x - vt) \quad \& \quad x = ct = \gamma(x' + vt')$$

$$\Rightarrow \quad ct' = \gamma(ct - vt) \quad \& \quad ct = \gamma(ct' + vt')$$

$$\therefore \quad ct = \gamma\, ct'\left(1 + \frac{v}{c}\right)$$

$$= \gamma\left(\gamma(ct - vt)\right)\left(1 - \frac{v}{c}\right) = \gamma^2 t\,(c - v)\left(1 + v/c\right)$$

$$\Rightarrow \quad ct = \gamma^2 t\left(\frac{1}{c}\right)\underbrace{(c - v)(c + v)}_{c^2 - v^2}$$

$$\Rightarrow \quad \gamma^2 = \frac{c^2}{c^2 - v^2}$$

$$\Rightarrow \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}$$

$$x = \gamma(x' - vt') \quad \& \quad x' = \gamma(x + vt)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}$$

Relationship between t & t':

$$x = \gamma(\underbrace{x'}_{= \gamma(x+vt)} - vt') = \gamma(\gamma x + \gamma v t - v t')$$

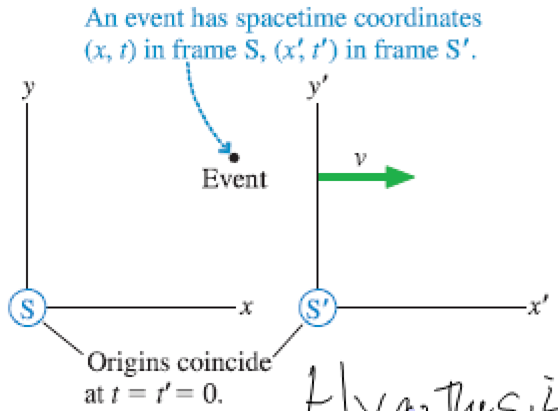
$$\Rightarrow \quad \gamma v t' = x(\gamma^2 - 1) + \gamma^2 v t$$

$$\Rightarrow \quad t' = \frac{x}{v} \frac{\gamma^2 - 1}{\gamma} + \gamma t = \gamma\left(t - \frac{\gamma^2-1}{\gamma^2}\left(\frac{x}{v}\right)\right)$$

$$\frac{\gamma^2 - 1}{\gamma^2} = 1 - \frac{1}{\gamma^2} = 1 - (1 - \beta^2) = \beta^2 = \left(\frac{v}{c}\right)^2$$

$$\therefore \quad t' = \gamma\left(t - \frac{v}{c^2} x\right)$$

Lorentz transformation



Event 1: A flash of light is emitted from the origin of both reference frames ($x = x' = 0$) at the instant they coincide ($t = t' = 0$).

Event 2: The light strikes a light detector. The spacetime coordinates of this event are (x, t) in frame S and (x', t') in frame S'.

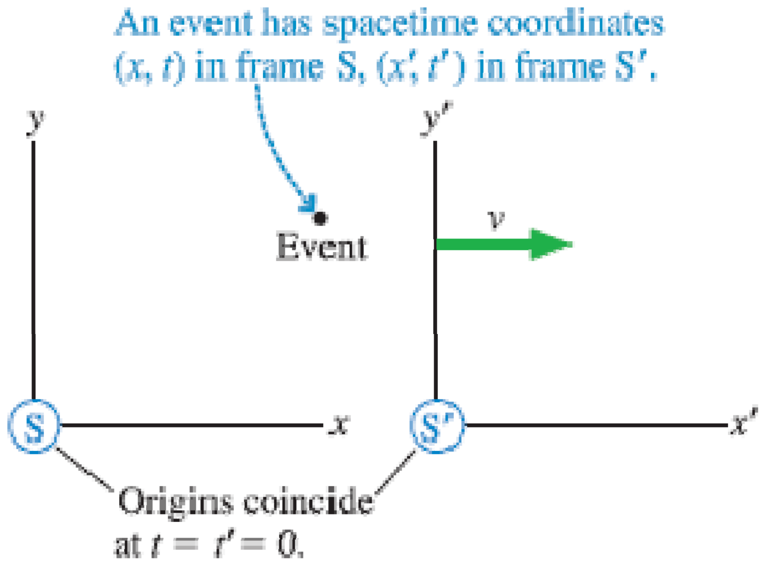
Hypothesis : $x' = \gamma(x - vt)$ & $x = \gamma(x' - vt')$

$\Rightarrow$ $x' = ct'$ = How far light travels to detector in frame S'

$x = ct$ = How far light travels to detector in frame S

Use Hypothesis to derive what γ has to be to satisfy "the speed of light is constant in all frames" condition.

Lorentz transformation



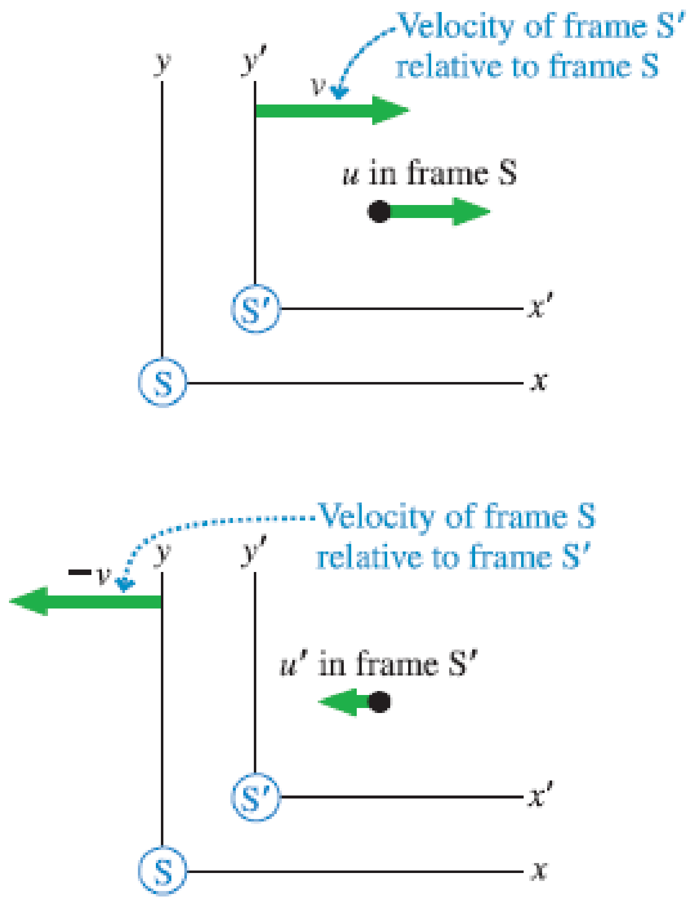
$$
\begin{aligned}
x' &= \gamma(x - vt) & \qquad x &= \gamma(x' + vt') \\
y' &= y & y &= y' \\
z' &= z & z &= z' \\
t' &= \gamma(t - vx/c^2) & t &= \gamma(t' + vx'/c^2)
\end{aligned}
$$

$$
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}
$$

The Lorentz transformations transform the spacetime coordinates of *one* event. Compare these to the Galilean transformation equations in Equations 37.1.

Lorentz velocity transformation

FIGURE 37.33 The velocity of a moving object is measured to be u in frame S and u' in frame S′.



$$u = \frac{dx}{dt} \quad \& \quad u' = \frac{dx'}{dt'}$$

Relationship between u & u'?

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\Rightarrow dx' = \gamma(dx - v\,dt), \quad dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$\therefore u' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

$$\Rightarrow u' = \frac{u - v}{1 - \frac{v}{c^2}u}$$

Similarly: $x = \gamma(x' + vt')$, $t = \gamma\left(t' + \frac{v}{c^2}x'\right)$

$$\Rightarrow u = \frac{u' + v}{1 + \frac{v}{c^2}u'}$$

Lorentz transformation

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$
$$y' = y \qquad y = y'$$
$$z' = z \qquad z = z'$$
$$t' = \gamma(t - vx/c^2) \qquad t = \gamma(t' + vx'/c^2)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

$$u' = \frac{u - v}{1 - uv/c^2} \quad \text{and} \quad u = \frac{u' + v}{1 + u'v/c^2}$$

Taylor Series:

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x-x_0) + \frac{1}{2}\left.\frac{d^2f}{dx^2}\right|_{x=x_0}(x-x_0)^2 + \cdots$$

Let $f(x) = (1 \pm x)^n$, & expand around $x_0 = 0$

$$f(x=0) = 1$$

$$\left.\frac{df}{dx}\right|_{x=0} = n(1 \pm x)^{n-1}(\pm 1) = \pm n$$

$$\Rightarrow \boxed{(1 \pm x)^n \approx 1 \pm nx}$$ for x close to $x_0 = 0$ or $x \ll 1$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2} \approx 1 + \frac{1}{2}\beta^2$$ for $\beta \ll 1$

$$\sqrt{1-\beta^2} = (1-\beta^2)^{1/2} \approx 1 - \frac{1}{2}\beta^2$$

Relativistic Momentum

Consider Elastic Collisions in frame S:

$$m_1 u_{1f} + m_2 u_{2f} = m_1 u_{1i} + m_2 u_{2i}$$

Does this relationship hold in all inertial reference frames?

In particular, Consider Reference frame S′

traveling at velocity V w.r.t. S

$$u = \frac{u' - v}{1 - \frac{v}{c^2} u'}$$

For Simplicity, assume $m_1 = m_2 = m$

$$u_{1f} + u_{2f} = u_{1i} + u_{2i}$$

Does $u_{1f}' + u_{2f}' = u_{1i}' + u_{2i}'$!?!?

$$u_{1f} + u_{2f} = u_{1i} + u_{2i} \qquad u = \frac{u' - v}{1 - \frac{v}{c^2} u'}$$

$$\Rightarrow \quad \frac{u'_{1f} - v}{1 - \frac{v}{c^2} u'_{1f}} + \frac{u'_{2f} - v}{1 - \frac{v}{c^2} u'_{2f}} = \frac{u'_{1i} - v}{1 - \frac{v}{c^2} u'_{1i}} + \frac{u'_{2i} - v}{1 - \frac{v}{c^2} u'_{2i}}$$

Choose S' such that $v = u'_{2i}$

Also, Drop primes & add them back later

$$\frac{u_{1f} - v}{1 - \frac{v}{c^2} u_{1f}} + \frac{u_{2f} - v}{1 - \underbrace{\frac{v}{c^2}}_{\alpha} u_{2f}} = \frac{u_{1i} - v}{1 - \frac{v}{c^2} u_{1i}}$$

$$(u_{1f} - v)(1 - \alpha u_{2f})(1 - \alpha u_{1i}) + (u_{2f} - v)(1 - \alpha u_{1f})(1 - \alpha u_{1i})$$
$$= (u_{1i} - v)(1 - \alpha u_{1f})(1 - \alpha u_{2f})$$

$$\frac{u_{1f}-v}{1-\frac{v}{c^2}u_{1f}} + \frac{u_{2f}-v}{1-\underbrace{\frac{v}{c^2}}_{\alpha}u_{2f}} = \frac{u_{1i}-v}{1-\frac{v}{c^2}u_{1i}}$$

$$(u_{1f}-v)(1-\alpha u_{2f})(1-\alpha u_{1i}) + (u_{2f}-v)(1-\alpha u_{1f})(1-\alpha u_{1i})$$
$$= (u_{1i}-v)(1-\alpha u_{1f})(1-\alpha u_{2f})$$

$$(u_{1f}-v)(1-2\alpha u_{1i}u_{2f}+\alpha^2 u_{1i}u_{2f}) + (u_{2f}-v)(1-2\alpha u_{1f}u_{1i}+\alpha^2 u_{1i}u_{1f})$$
$$= (u_{1i}-v)(1-2\alpha u_{1f}u_{2f}+\alpha^2 u_{1f}u_{2f})$$

$$u_{1f}+u_{2f} + (u_{1f}-v)(\alpha^2 u_{1i}u_{2f} - 2\alpha u_{1i}u_{2f}) + (u_{2f}-v)(\alpha^2 u_{1i}u_{1f} - 2\alpha u_{1f}u_{1i})$$
$$= u_{1i} + (u_{1i}-v)(\alpha^2 u_{1f}u_{2f} - 2\alpha u_{1f}u_{2f})$$

$\Rightarrow u'_{1f} + u'_{2f} - u'_{1i} =$ A Bunch of stuff that is not Zero

$\Rightarrow u'_{1f} + u'_{2f} - u'_{1i} =$ A Bunch of stuff that is not zero

$\therefore$ Using $m_1 u_{1f} + m_2 u_{2f} = m_1 u_{1i} + m_2 u_{2i}$

Does not work in all inertial reference frames

It only happens to work in reference frames where all particles velocities $<< c$

Recall, we started with conservation of momentum in frame S:

$$m u_{1f} + m u_{2f} - (m u_{1i} + m u_{2i}) = 0$$

And we end up with a result in frame S'

$$m u'_{1f} + m u'_{2f} - (m u'_{1i} + m u'_{2i}) \neq 0$$

Relativistic Momentum

We either (a) abandon the concept of conservation of momentum

or (b) redefine concept of momentum so that some law works in a similar manner + reduces to usual law when $\frac{v}{c} << 1$

Relativistic Momentum

The following is not a proof, but gives some connection to the new momentum definition:

$$p = \gamma_p m u \quad \text{where} \quad \gamma_p = \frac{1}{\sqrt{1 - u^2/c^2}}$$

$p = mu = m \dfrac{\Delta x}{\Delta t}$ where normally Δx + Δt are measured in a reference frame.

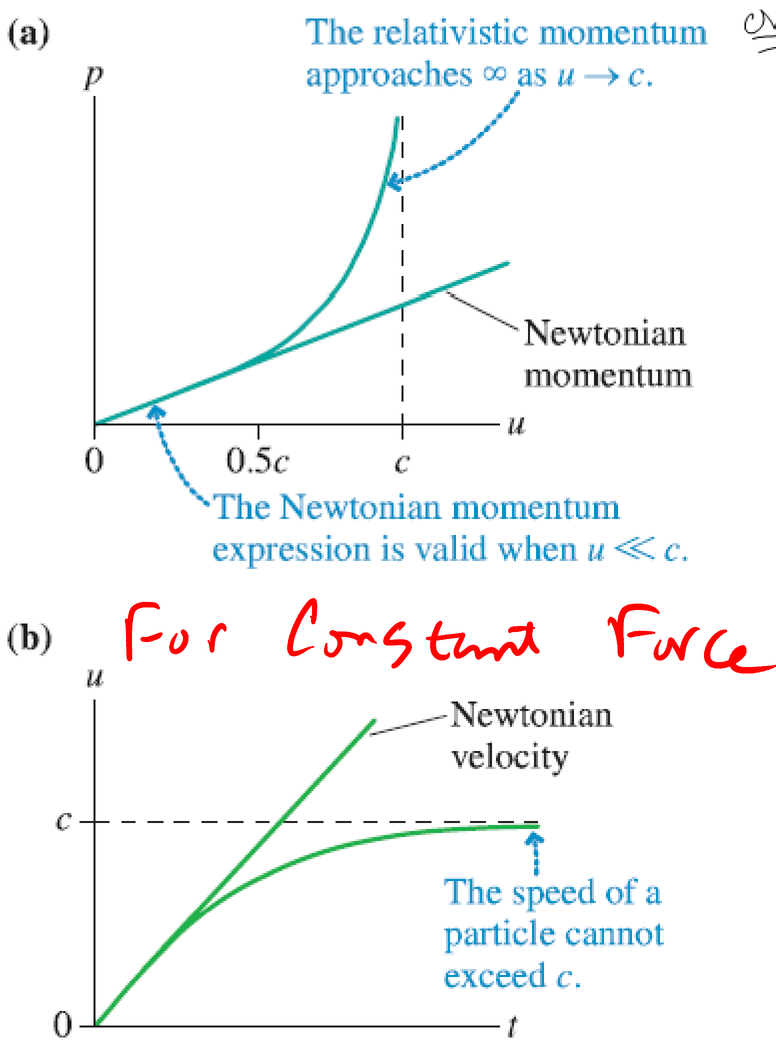
Both Δx & Δt change depending on the reference frame.

What if we use $\Delta t \to \Delta \tau$, proper time or the clock in the particle's rest frame, in the definition?

$$\Rightarrow p = m \frac{\Delta x}{\Delta \tau} = m \frac{\Delta x}{\Delta t \sqrt{1 - (u/c)^2}} = \gamma_p m u$$

Relativistic Momentum

FIGURE 37.34 The speed of a particle cannot reach the speed of light.



$p = \gamma_p m u$ where $\gamma_p = \dfrac{1}{\sqrt{1 - u^2/c^2}}$

or

$p = m_{eff}\, u$ where $m_{eff} = \dfrac{m_0}{\sqrt{1 - u^2/c^2}}$

& m_0 = Rest mass

For Constant Force,

$p = F \cdot t$

$m_{eff}\, u = F t$

Causality and Information Flow

A word about causality:

One event which <u>causes</u> another can not communicate faster than the speed of Light

Caveat: Information can not flow faster than the speed of Light

Relativistic Energy

Conservation of energy in Newtonian dynamics:

$$KE = \frac{p^2}{2m} \quad ; \quad E = KE + PE$$

We would expect the law of conservation of energy to not hold since it involves the momentum.

We will derive a new law that reduces to the old one in the limit of small velocity

Relativistic Energy

Let a particle of mass m move through distance Δx during a time interval Δt, as measured in reference frame S. The spacetime interval is

$$s^2 = c^2(\Delta t)^2 - (\Delta x)^2 = \text{invariant}$$

We can turn this into an expression involving momentum if we multiply by $(m/\Delta\tau)^2$, where $\Delta\tau$ is the proper time (i.e., the time measured by the particle). Doing so gives

$$(mc)^2\left(\frac{\Delta t}{\Delta\tau}\right)^2 - \left(\frac{m\Delta x}{\Delta\tau}\right)^2 = (mc)^2\left(\frac{\Delta t}{\Delta\tau}\right)^2 - p^2 = \text{invariant} \qquad (37.37)$$

where we used $p = m(\Delta x/\Delta\tau)$ from Equation 37.32.

Now Δt, the time interval in frame S, is related to the proper time by the time-dilation result $\Delta t = \gamma_p \Delta\tau$. With this change, Equation 37.37 becomes

$$(\gamma_p mc)^2 - p^2 = \text{invariant}$$

Finally, for reasons that will be clear in a minute, we multiply by c^2, to get

$$(\gamma_p mc^2)^2 - (pc)^2 = \text{invariant} \qquad (37.38)$$

Relativistic Energy

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$$(mc)^2\left(\frac{\Delta t}{\Delta\tau}\right)^2 - \left(\frac{m\Delta x}{\Delta\tau}\right)^2 = (mc)^2\left(\frac{\Delta t}{\Delta\tau}\right)^2 - p^2 = \text{invariant} \qquad (37.37)$$

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Relativistic Energy

$$(\gamma_p mc^2)^2 - (pc)^2 = \text{invariant}$$

$$\underbrace{(\gamma_p mc^2)^2 - (pc)^2}_{\text{frame } S} = \underbrace{(\gamma_p' mc^2)^2 - (p'c)^2}_{\text{frame } S'}$$

$$(\gamma_p mc^2)^2 - (pc)^2 = \underbrace{(mc^2)^2}_{\text{"Particle at Rest" frame}}$$

"Particle at Rest" frame

$(p' = 0 \Rightarrow \gamma_p' = 1)$

Relativistic Energy

$$(\gamma_p mc^2)^2 - (pc)^2 = \text{invariant}$$

$$\gamma_p mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \approx \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right)mc^2 = mc^2 + \underbrace{\frac{1}{2}mu^2}_{\text{KE}}$$

u<<c

New!

An inherent energy associated with the particles rest mass!

Call it "rest" Energy of particle

$$(\gamma_p mc^2)^2 - (pc)^2 = (mc^2)^2$$

Relativistic Energy

$$\gamma_p mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \approx \left(1 + \frac{1}{2}\frac{u^2}{c^2}\right)mc^2 = mc^2 + \frac{1}{2}mu^2$$

Define E, KE, & rest Energy E_0

$$E = \gamma_p mc^2 = E_0 + K = \text{rest energy} + \text{kinetic energy}$$

This total energy consists of a **rest energy**

$$E_0 = mc^2$$

and a relativistic expression for the *kinetic energy*

$$K = (\gamma_p - 1)mc^2 = (\gamma_p - 1)E_0$$

$$E^2 - (pc)^2 = E_0^2$$

$$\underbrace{E}_{\gamma_p mc^2}{}^2 - (pc)^2 = \underbrace{E_0}_{mc^2}{}^2$$

$$pc = (\gamma_p mu)c = \frac{u}{c}\underbrace{(\gamma_p mc^2)}_{E}$$

$$\therefore\ E^2 - (pc)^2 = E_0^2$$

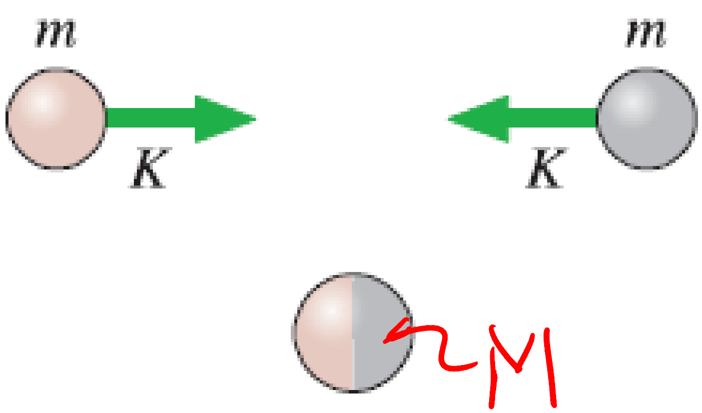
$$\Rightarrow \underbrace{E^2}_{(\gamma_p mc^2)^2}\underbrace{\left(1 - \left(\frac{u}{c}\right)^2\right)}_{(1/\gamma_p)^2} = \underbrace{E_0^2}_{(mc^2)^2}$$

✓ Consistent

Relativistic Energy

Law of conservation of total energy The energy $E = \sum E_i$ of an isolated system is conserved, where $E_i = (\gamma_p)_i m_i c^2$ is the total energy of particle i.

FIGURE 37.38 An inelastic collision between two balls of clay does not seem to conserve the total energy E.



$E_i = 2mc^2 + 2K$

$E_f = 2mc^2$? X

$$E_f = E_i \Rightarrow 2Mc^2 = 2mc^2 + 2K$$

$$\Rightarrow M > m \quad !!$$

Mass <u>created</u> <u>from</u> <u>KE</u>

Can also have
photons → particles
& particles → KE,
photons

For Marketing Majors only!

FIGURE 37.37 The velocity-energy-momentum triangle.