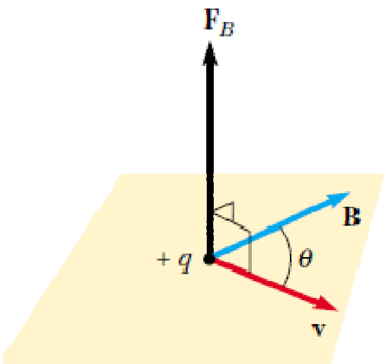
MIT visualizations:
Biot Savart Law,
Integrating a circular current loop on axis

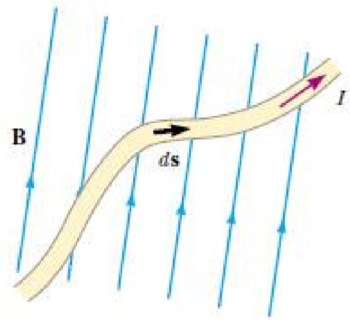
Last time:

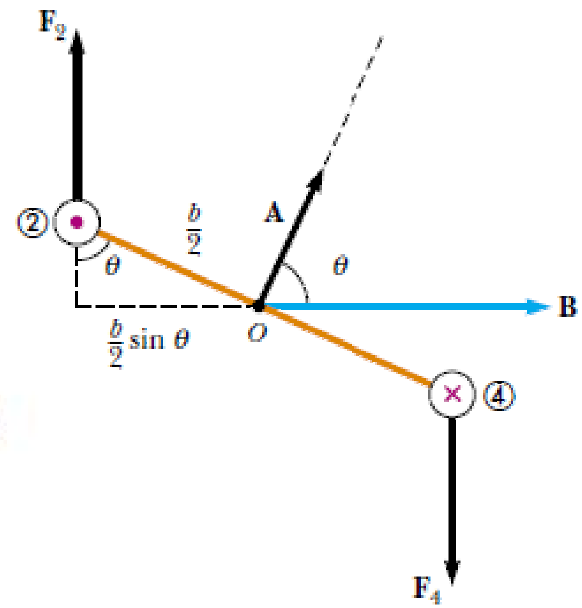
-Moving charges (and currents) create magnetic fields

-Moving charges (and currents) feel a force in magnetic fields

- $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$ Defines the B-field; Examples - cyclotron frequency, velocity selector, mass spectrometer







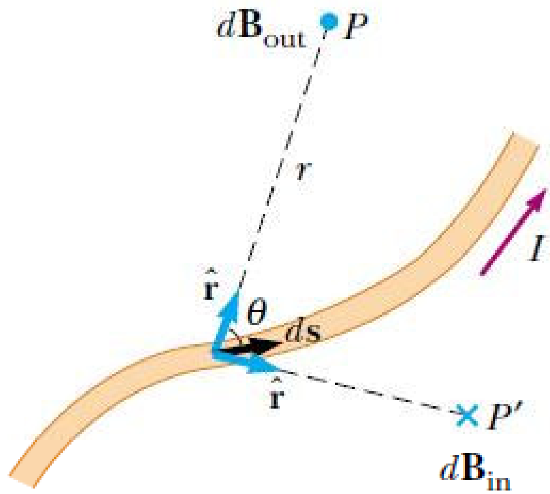
- Force on a wire segment: $$\mathbf{F}_B = I\int_a^b d\mathbf{s} \times \mathbf{B}$$

- Force on a straight wire segment in uniform B-field: $$\mathbf{F}_B = I\mathbf{L} \times \mathbf{B}$$

- Torque on current loop in uniform magnetic field: $$\boldsymbol{\mu} = I\mathbf{A} \qquad \boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$$

- Potential energy of magnetic dipole (current loop) in magnetic field $$U = -\boldsymbol{\mu}\cdot\mathbf{B}$$

The Source of the Magnetic Field: Moving Charges



Biot–Savart law

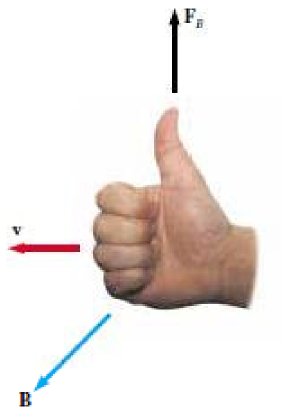
$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{s} \times \hat{\mathbf{r}}}{r^2}$$

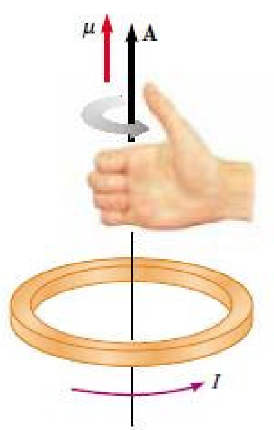
- The vector $d\mathbf{B}$ is perpendicular both to $d\mathbf{s}$ (which points in the direction of the current) and to the unit vector $\hat{\mathbf{r}}$ directed from $d\mathbf{s}$ toward P.
- The magnitude of $d\mathbf{B}$ is inversely proportional to r^2, where r is the distance from $d\mathbf{s}$ to P.
- The magnitude of $d\mathbf{B}$ is proportional to the current and to the magnitude ds of the length element $d\mathbf{s}$.
- The magnitude of $d\mathbf{B}$ is proportional to $\sin\theta$, where θ is the angle between the vectors $d\mathbf{s}$ and $\hat{\mathbf{r}}$.

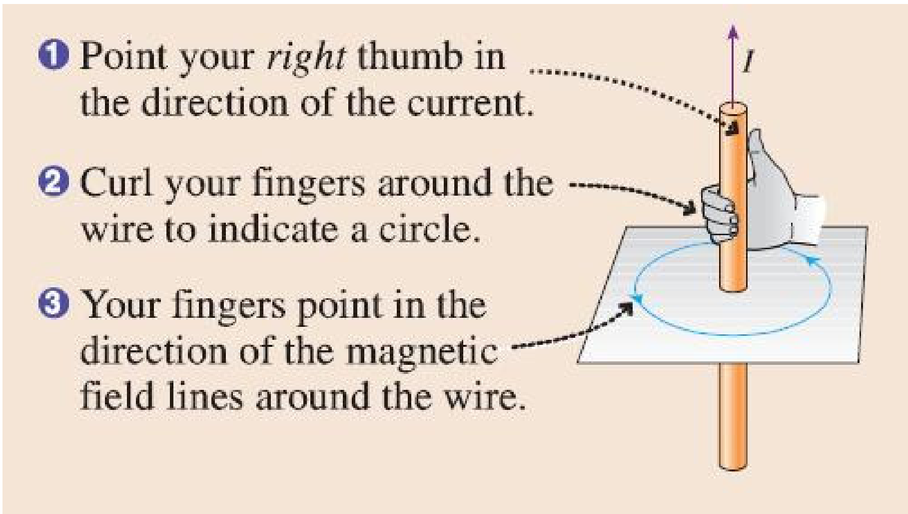
MIT Biot Savart visualization

http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/magnetostatics/03-CurrentElement3d/CurrentElement3d.htm

Summary of three Right Hand Rules:







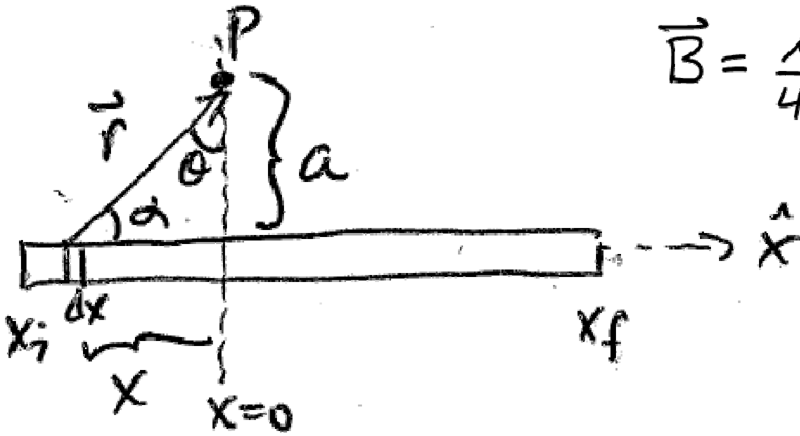
Applying the Biot-Savart Law

Arbitrary shaped currents difficult to calculate

Simple cases, one can solve relatively easily with pen and paper:

current loops, straight wire segments

Last time, we found B-field for straight wire segment with current in x- direction:

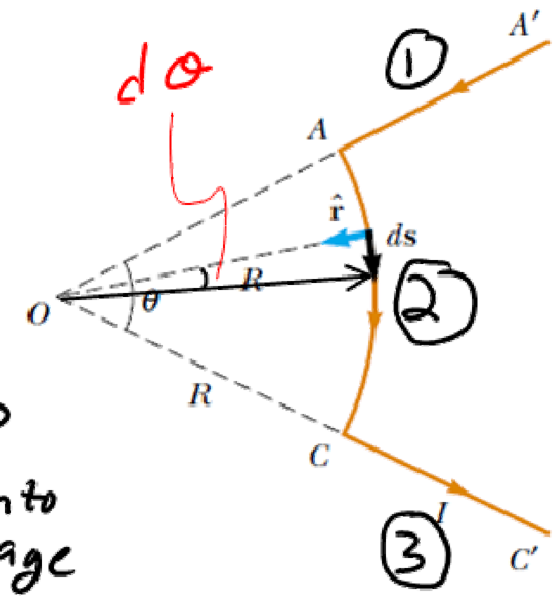


$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I}{a}\left[\frac{x_f}{\sqrt{x_f^2+a^2}} - \frac{x_i}{\sqrt{x_i^2+a^2}}\right]\hat{z}$$

And limit xf=-xi=L-> infinity for an infinitely long straight wire:

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{a}\hat{z}$$

Applying the Biot-Savart Law: Circular arc → Circle



$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{s} \times \hat{r}}{r^2}$$

For ① + ③, $d\vec{s} \parallel \hat{r} \Rightarrow d\vec{s} \times \hat{r} = 0$

For ②, $d\vec{s} \perp \hat{r} \Rightarrow d\vec{s} \times \hat{r} = ds$ into page

$ds = a\, d\theta$

$$|\vec{B}| = \int_{②} |d\vec{B}| = \frac{\mu_0}{4\pi} I \int_0^{\theta} \frac{a\, d\theta}{a^2} = \frac{\mu_0}{4\pi} \frac{I}{a} \theta$$

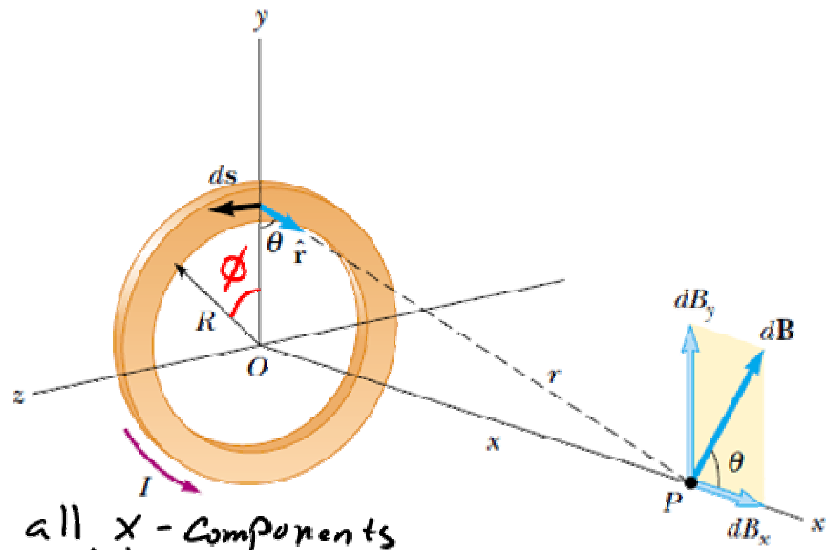
For Full Circle: $\theta \rightarrow 2\pi$

$$\vec{B}_{center} = \frac{\mu_0 I}{2a} \quad \text{into page}$$

http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/magnetostatics/08-RingMagInt/MagRingIntFullScreen.htm

http://ocw.mit.edu/ans7870/8/8.02T/f04/visualizations/magnetostatics/09-RingMagField/RingMagFieldFullScreen.htm

Applying the Biot-Savart Law:
On-axis of circular loop



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{s} \times \hat{r}}{r^2}$$

When we integrate, $B_y \to 0$ & all x-components add

Let θ be angle between $d\vec{B}$ & $\hat{x}$:

$$|d\vec{B}| = d\vec{B} \cdot \hat{x} = |dB| \cos\theta$$

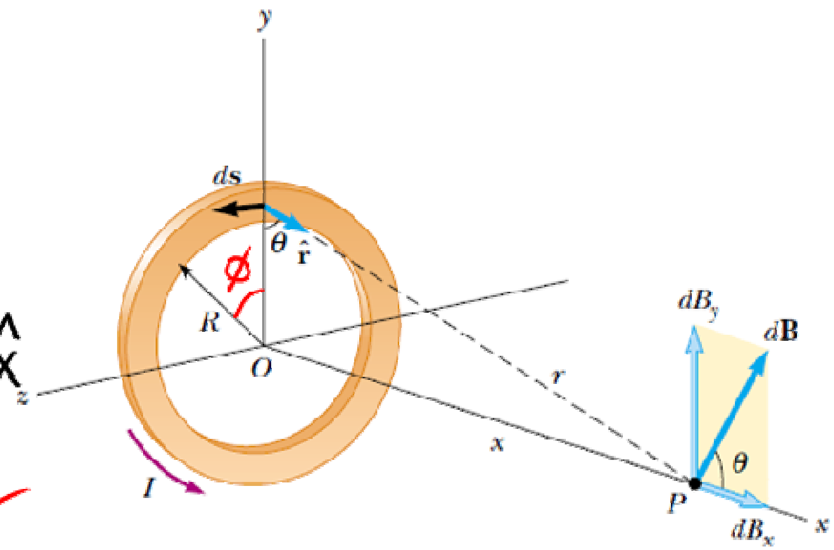
$$\cos\theta = \frac{R}{r}, \quad r = \sqrt{x^2 + R^2}, \quad d\vec{s} \perp \hat{r}$$

$$\Rightarrow |d\vec{s}| = R\, d\phi$$

$$|\vec{B}| = \int |dB| = \frac{\mu_0}{4\pi} I \int \frac{R\, d\phi}{r^2} \frac{R}{r}$$

$$\vec{B} = \frac{\mu_0}{2} I \frac{R^2}{(x^2+R^2)^{3/2}} \hat{x}$$

Applying the Biot-Savart Law: On-axis of circular loop



$$\vec{B} = \frac{\mu_0}{2} I \frac{R^2}{(x^2+R^2)^{3/2}} \hat{x}$$

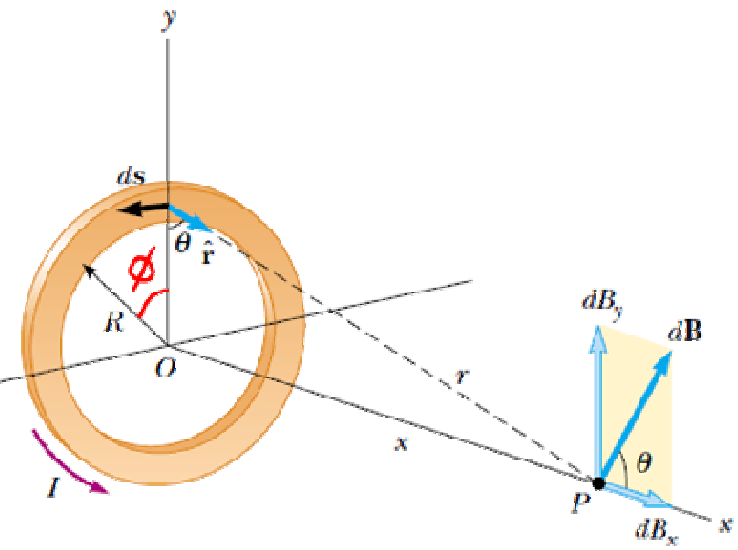
$$\lim_{x \to 0} \vec{B} = \frac{\mu_0 I}{2R} \hat{x}$$

Let $x \gg R$, & $(x^2+R^2)^{3/2} = x^3 \underbrace{\left(1+\frac{R^2}{x^2}\right)^{3/2}}_{\to 1}$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} 2 \frac{\overbrace{I \pi R^2 \hat{x}}^{\equiv \vec{\mu} = I\vec{A}}}{x^3}$$

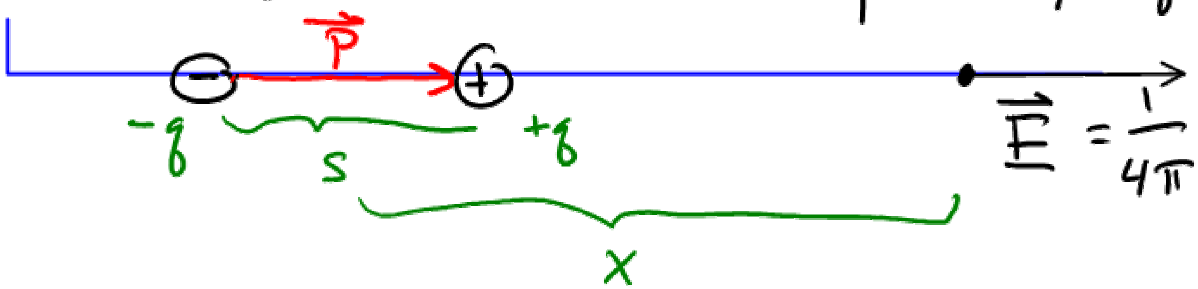
$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \frac{2\vec{\mu}}{x^3} \quad \text{on axis}$$

Applying the Biot-Savart Law: On-axis of circular loop



$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \frac{2\vec{\mu}}{x^3} \quad \text{on axis}$$

Looks Like Electric dipole: $\vec{p} \equiv q\vec{s}$



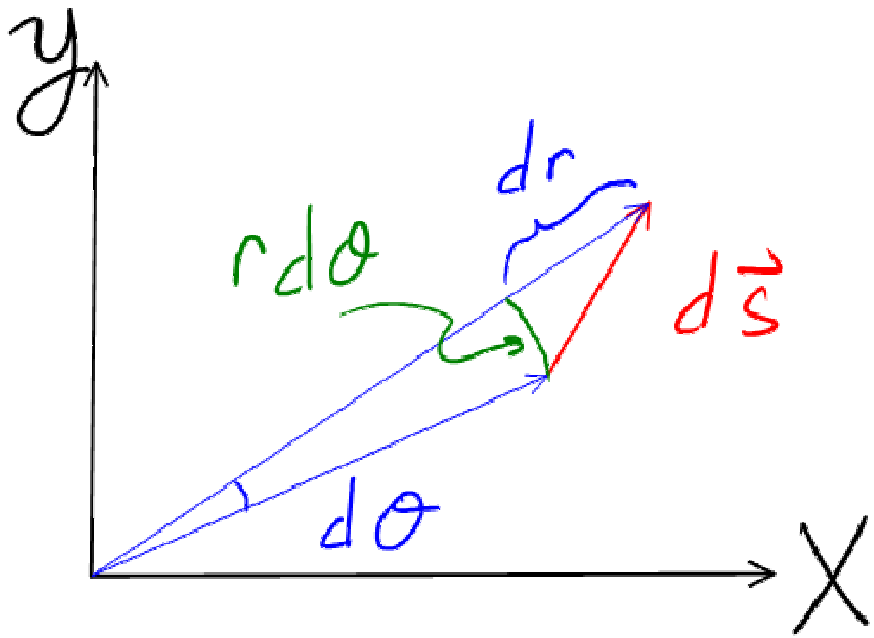
$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\vec{p}}{x^3}$$

Also Recall, $\vec{\tau}_B = \vec{\mu} \times \vec{B}$ $\qquad \vec{\tau}_E = \vec{p} \times \vec{E}$

$\mathcal{U}_B = -\vec{\mu} \cdot \vec{B}$ $\qquad \mathcal{U}_E = -\vec{p} \cdot \vec{E}$

Current Loops are "magnetic Dipoles"

Ampere's Law: An easier way to find B-fields
(in very special circumstances)



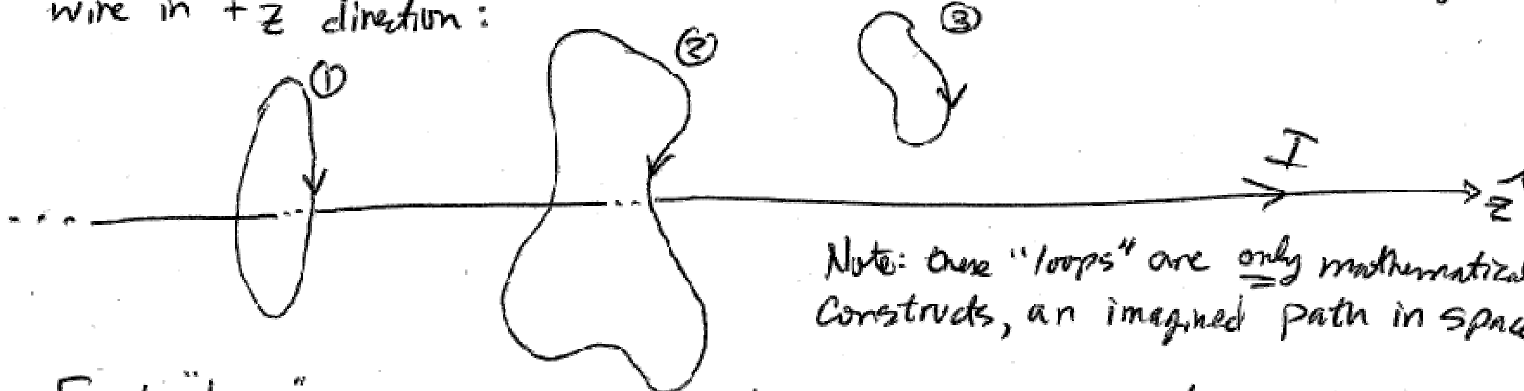
$$d\vec{s} = r\,d\theta\,\hat{\theta} + dr\,\hat{r}$$

$$\hat{\theta} \times \hat{r} = -\hat{r} \times \hat{\theta} = +\hat{z}$$

(Note: $\hat{r}$ is from current segment to origin)

Ampere's Law: An easier way to find B-fields
(in very special circumstances)

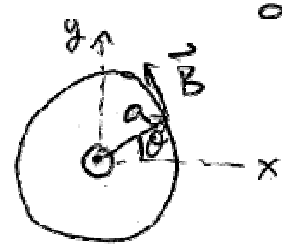
Consider the following cases for an ∞-ite straight current carrying wire in $+\hat{z}$ direction:



Note: these "loops" are only mathematical constructs, an imagined path in space

Each "loop" is in the xy-plane. We will evaluate $\int \vec{B}\cdot d\vec{s}$ around the closed loops for cases ①, ②, & ③ for an ∞-ite straight current carrying wire

①



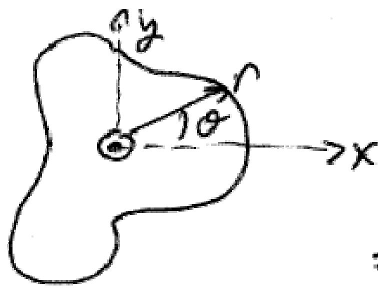
$$\vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\theta}, \quad d\vec{s} = \underbrace{r}_{a}\, d\theta\, \hat{\theta} + \underbrace{dr}_{0,\ \text{constant radius}}\hat{r}$$

$$\oint \vec{B}\cdot d\vec{s} = \oint \frac{\mu_0 I}{2\pi a}\left[\underbrace{a\,d\theta}_{2\pi}\ \underbrace{\hat{\theta}\cdot\hat{\theta}}_{1} + \underbrace{\hat{\theta}\cdot\hat{r}}_{0}\ \underbrace{dr}_{0}\right]$$

$$\Rightarrow \boxed{\oint_{①} \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{through}}}$$

Ampere's Law: An easier way to find B-fields
(in very special circumstances)

②



$$\oint \vec{B}\cdot d\vec{s} = \oint \left[\frac{\mu_0 I}{2\pi r}\hat{\theta}\right]\cdot\left[r\,d\theta\,\hat{\theta} + dr\,\hat{r}\right]$$

$$= \frac{\mu_0 I}{2\pi}\oint\left[d\theta\,\underbrace{\hat{\theta}\cdot\hat{\theta}}_{=1} + \frac{dr}{r}\,\underbrace{\hat{\theta}\cdot\hat{r}}_{=0 \text{ since } \hat{\theta}\perp\hat{r}}\right]$$

r changes as we go around loop, but drops out

$$\Rightarrow \oint_{②} \vec{B}\cdot d\vec{s} = \mu_0 I_{through}$$

③

integrate around loop & note that $\theta_i = \theta_f$!

$$\oint \vec{B}\cdot d\vec{s} = \int_{\theta_i}^{\theta_f}\left(\frac{\mu_0 I}{2\pi}\right)\left[d\theta\,\underbrace{\hat{\theta}\cdot\hat{\theta}}_{1} + \frac{dr}{r}\,\underbrace{\hat{\theta}\cdot\hat{r}}_{0,\ \hat{\theta}\perp\hat{r}}\right]$$

$$= \frac{\mu_0 I}{2\pi}\left(\theta_f - \theta_i\right) = 0$$

$$\therefore \oint \vec{B}\cdot d\vec{s} = 0$$, no current through loop.

Ampere's Law: An easier way to find B-fields (in very special circumstances)

For any arbitrary loop (not just 2-D loops!):

Ampere's Law $$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{through}$$

Use Ampere's Law to find B-field from current (in very special circumstances)

1. Current that does not go through "Amperian Loop" does not contribute to the integral

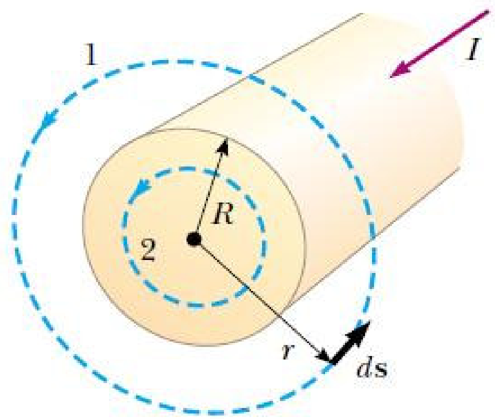
2. Current through is the "net" current through loop

3. Try to choose loops where B-field is either parallel or perpendicular to ds, the Amperian loop. To do this, remember that symmetry is your friend!

Review pages 850-853!!

Ampere's Law: Example, Finite size infinite wire

Calculate the B-field everywhere from a finite size, straight, infinite wire with uniform current.

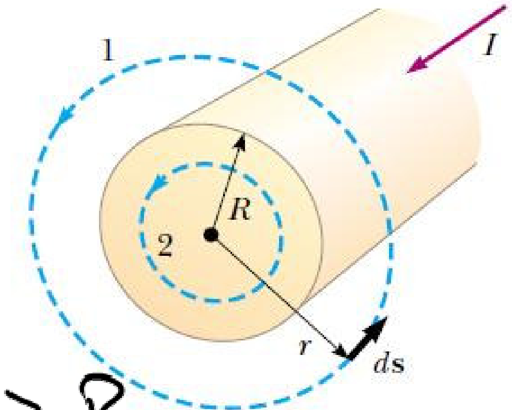


From symmetry,
Br=0 (reverse current and flip cylinder)
Bz=0 (B=0 at infinity, Amperian rectangular loop from infinity parallel to axial direction implies zero Bz everywhere)

Only azimuthal component exists. Therefore, amperian loops are circles such that B parallel to ds.

Ampere's Law: Example, Finite size infinite wire

Calculate the B-field everywhere from a finite size, straight, infinite wire with uniform current.



Case I: outside of wire, $r \geq R$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$I_{enc} = I$, total current in wire

$\vec{B} \parallel d\vec{s}$

$$\vec{B} \cdot d\vec{s} = B\,ds = B\,r\,d\theta$$

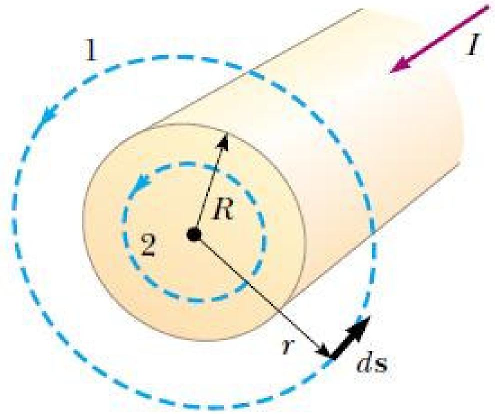
$$\oint \vec{B} \cdot d\vec{s} = \int_0^{2\pi} B\,r\,d\theta = \underbrace{2\pi r}_{\text{Circumference}} B = \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0}{2\pi} \frac{I}{r}$$

direction from RHR

Ampere's Law: Example, Finite size infinite wire

Calculate the B-field everywhere from a finite size, straight, infinite wire with uniform current.



Case II: $r \leq R$ inside wire

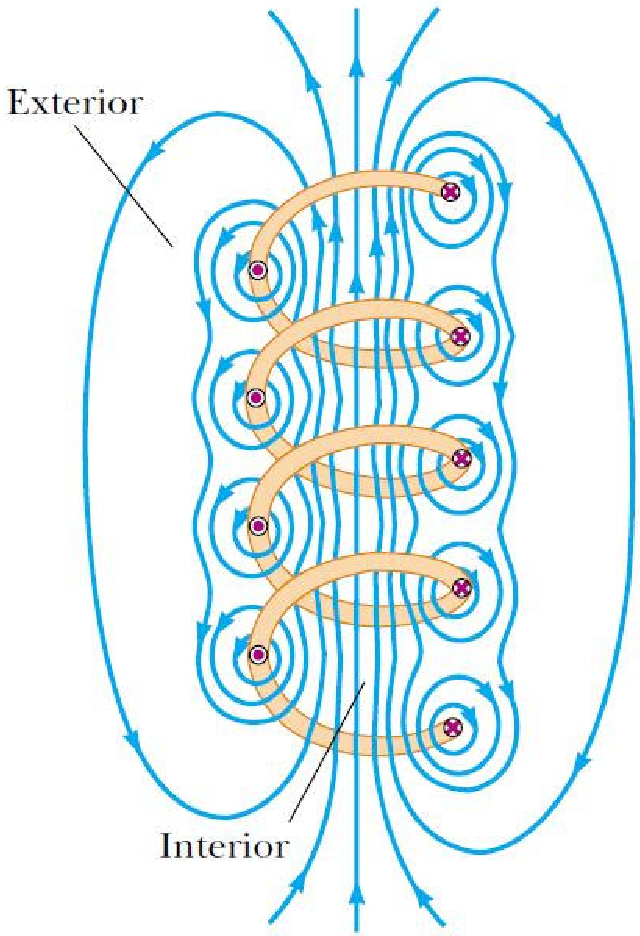
$$I_{enc} = \frac{\pi r^2}{\pi R^3} I$$

$$\oint \vec{B} \cdot d\vec{s} = \int_0^{2\pi} B r \, d\theta = 2\pi r B = \mu_0 I_{enc}$$

$$B = \frac{\mu_0 I}{2\pi} \frac{r}{R^2}$$

direction from RHR

Ampere's Law: Example, Infinitely long solenoid



As coils become more closely spaced, and the wires become thinner, and the length becomes much longer than the radius,

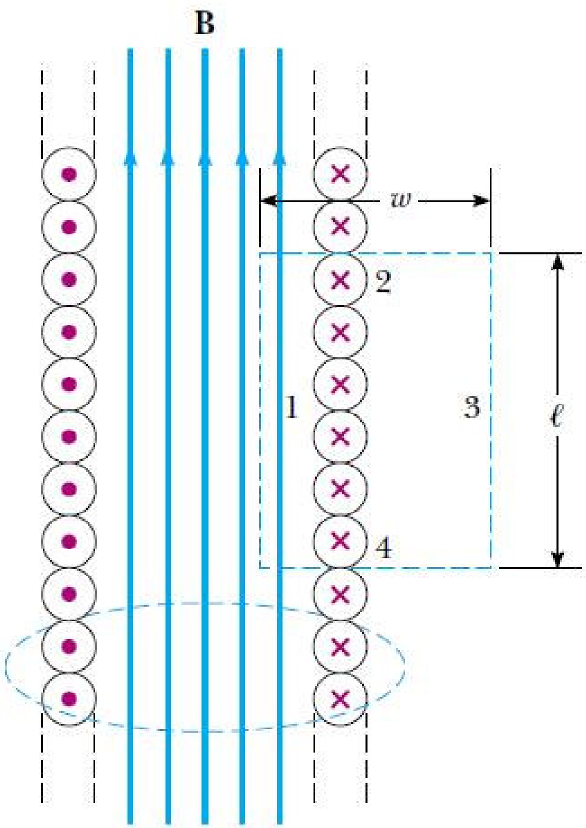
1. The B-field outside becomes very, very small (not at the ends, but away from sides)
2. The B-field inside points along the axial direction of the cylinder

Symmetry arguments for a sheet of current around a long cylinder:

1. Br=0 – time reversal + flipping cylinder, but time reversal would flip Br!
2. Azimuthal component? No, since if we choose Amperian loop perpendicular to axis, no current pierces it.
3. B=0 everywhere outside: any Amperian loop outside has zero current through it

Ampere's Law: Example, Infinitely long solenoid

Cross sectional view:



$$\oint \vec{B}\cdot d\vec{s} = \int_{①} \overbrace{\vec{B}\cdot d\vec{s}}^{\vec{B} \parallel ds,\ B\ell} + \int_{②} \overbrace{\vec{B}\cdot d\vec{s}}^{\vec{B} \perp d\vec{s},\ 0}$$

$$+ \int_{③} \underbrace{\vec{B}\cdot d\vec{s}}_{B=0 \text{ outside}} + \int_{④} \underbrace{\vec{B}\cdot d\vec{s}}_{\vec{B} \perp d\vec{s},\ 0}$$

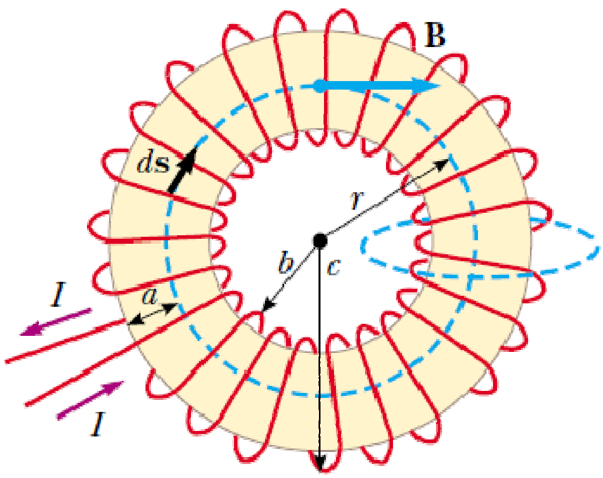
$I_{enc} = \frac{N}{L} \ell I$, N = # of turns
L = total length of Solenoid
I = Current in wire

$$B\ell = \mu_0 \left(\frac{N}{L} \ell I \right) \Rightarrow B = \mu_0 n I, \text{ RHR}$$

where n = # of turns per length

B is uniform inside solenoid!!

Ampere's Law: Example, Toroid



Solenoid bent in shape of donut.
B is circumferential.

∴ Draw amperian loop as circle inside torus
$\Rightarrow \vec{B} \parallel d\vec{s}$

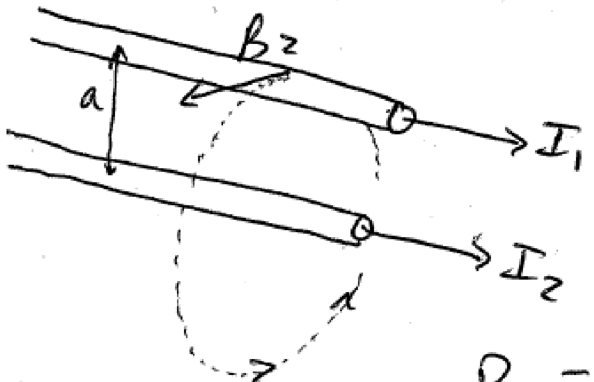
$$\oint \vec{B}\cdot d\vec{s} = B\cdot 2\pi r = \mu_0 (NI)$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 N I}{2\pi r}\,\hat{\theta}}$$

Force Between two parallel, straight current carrying wires:

- one wire produces a B-field @ location of other wire.

Consider 2 parallel wires carrying current:



I_2 produces B_2, & current I_1 feels a force. Let l_1 be the length of l_1 & wire 2 is infinitely long

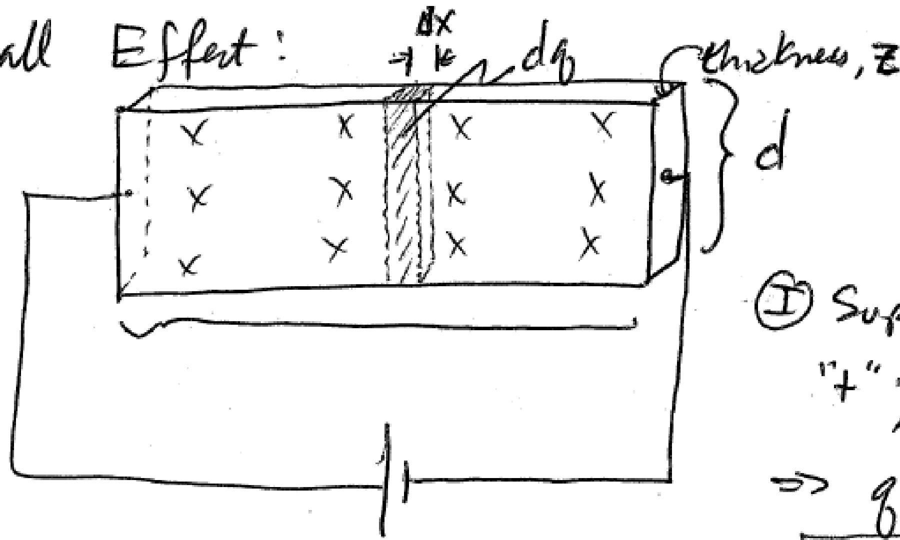
$$\Rightarrow \vec{F}_1 = I_1 \vec{l}_1 \times \vec{B}_2 = I_1 l_1 B_2 \text{ towards wire 2}$$

$$B_2 = \frac{\mu_0 I_2}{2\pi a}$$

$$\Rightarrow F_1 = I_1 l_1 \frac{\mu_0 I_2}{2\pi a} = \boxed{\frac{\mu_0}{2\pi a} I_1 I_2} \text{ towards wire 2}$$

Parallel currents attract, Opposite currents repel.

• Hall Effect:



Connect a Battery to Conducting Bar in magnetic field pointing into page.

(I) Suppose charge carriers are "+"; $\vec{V}_D \parallel \vec{I}$ (to the right)

$\Rightarrow$ $q\vec{V}_D \times \vec{B}$ up $\Rightarrow$ "+" charge carriers collect along top until $\vec{E}$ force equals $\vec{B}$ force:

$$q V_D B = q E_H \,, \quad E_H = \frac{\Delta V_H}{d} \quad \Rightarrow \Delta V_H = d\, V_D\, B$$

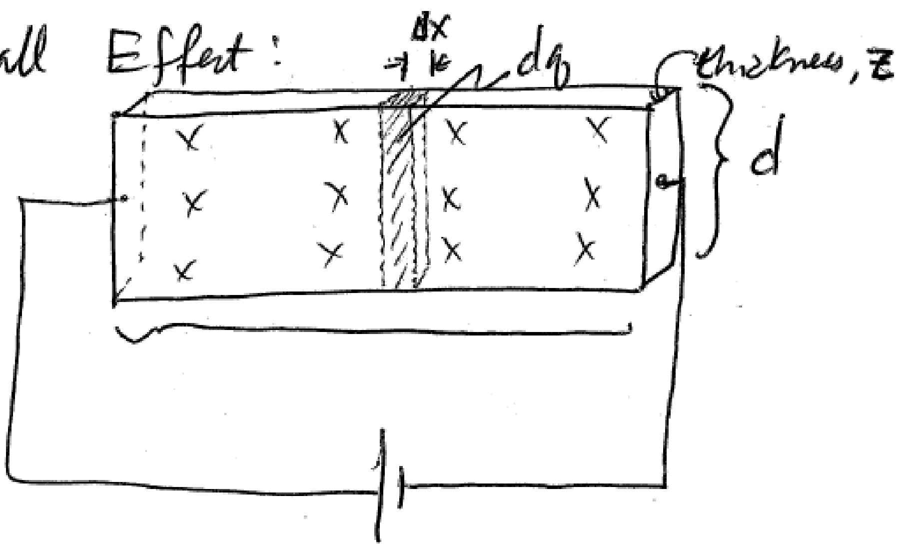
dq = amt of charge in volume element, $A_\perp$ = cross sectional area = $d \cdot Z$

$dx \equiv V_D\, dt$ $\ni$ all dq will move out of volume element in time dt

$$dq = \underset{\text{density of carriers}}{n}\; \underset{\text{charge of carrier}}{q_0}\; \underbrace{(A_\perp\, dx)}_{\text{volume}} = n q_0\, d Z\, V_D\, dt \Rightarrow \frac{dq}{dt} = I = n q_0\, d Z\, V_D \cancel{dt}$$

or $\Delta V_H = B \cdot d \left[\frac{I}{n q_0\, d Z} \right] = \boxed{\frac{IB}{n q_0 Z}}$

• Hall Effect:



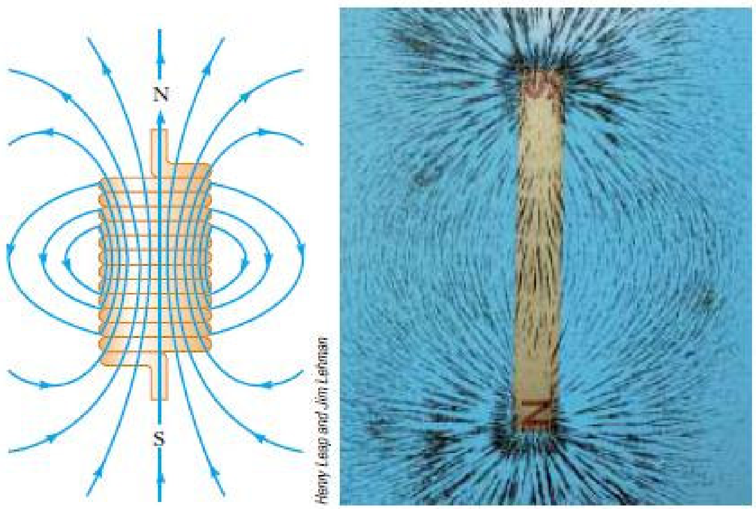
(II) Suppose charge carriers "−"; $\vec{V}_D$ is in opposite direction of current (to left)

$\Rightarrow$ $\underbrace{-|q|}_{\text{negative charge}} \; \underbrace{V_D \times B}_{\text{down}} = q V_D B$ up (negative down = up)

$\therefore$ $\boxed{\text{"−" charges will Build up on top}}$

Permanent magnets related to (tiny) currents:

Current loops look like magnets, and vice versa:



Permanent magnets can be thought of as a many tiny current loops created by the ‘spin’ of the electron. These tiny current loops (magnetic moments) tend to line up creating a macroscopic, large magnetic field. Note that, at least from a classical point of view, a charged sphere spinning creates a circulating current → magnetic field.