HW 13
Chapter 41: One-Dimensional Quantum Mechanics

▪ Conceptual Questions

41.2. Five.

41.4. $\eta_a = \eta_b > \eta_c$. η is independent of L. $\eta = \dfrac{\hbar}{\sqrt{2m(U_0 - E)}}$.

$$\eta_a = \eta_b = \frac{\hbar}{\sqrt{2m(10\text{ eV} - 5\text{ eV})}} \qquad \eta_c = \frac{\hbar}{\sqrt{2m(16\text{ eV} - 10\text{ eV})}}$$

▪ Exercises and Problems

41.1. Model: Model the electron as a particle in a rigid one-dimensional box of length L.
Solve: Absorption occurs from the ground state $n = 1$. It's reasonable to assume that the transition is from $n = 1$ to $n = 2$. The energy levels of an electron in a rigid box are

$$E_n = n^2 \frac{h^2}{8mL^2}$$

The absorbed photons must have just the right energy, so

$$E_{\text{ph}} = hf = \frac{hc}{\lambda} = \Delta E_{\text{elec}} = E_2 - E_1 = \frac{3h^2}{8mL^2}$$

$$\Rightarrow L = \sqrt{\frac{3h\lambda}{8mc}} = \sqrt{\frac{3(6.63\times10^{-34}\text{ J s})(6.00\times10^{-7}\text{ m})}{8(9.11\times10^{-31}\text{ kg})(3.0\times10^{8}\text{ m/s})}} = 7.39\times10^{-10}\text{ m} = 0.739\text{ nm}$$

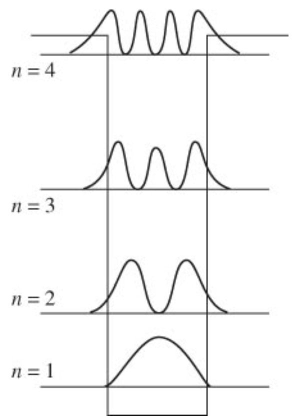
41.4. Model: Model the electron as a particle in a rigid one-dimensional box of length L.
Solve: From Equation 41.23, the energies of the stationary states for a particle in a box are $E_n = n^2E_1$, where E_n is the energy of the stationary state with *quantum number n*. It can be seen either from Figure 41.7 or from the wave function equation $\psi_n(x) = A\sin(n\pi x/L)$ that the wave function given in Figure Ex41.4 corresponds to n = 4. Thus,

$$E_4 = 16E \Rightarrow E_1 = \frac{E_4}{16} = \frac{12.0\text{ eV}}{16} = 0.75\text{ eV}$$

41.5. **Solve:** From Equation 41.41, the units of the penetration distance are

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} \Rightarrow \frac{\text{J}\times\text{s}}{\sqrt{\text{kg}\times\text{J}}} = \frac{\left(\text{kg}\times\text{m}^2/\text{s}^2\right)\times\text{s}}{\sqrt{\text{kg}\times\text{kg m}^2/\text{s}^2}} = \frac{\text{kg}\times\text{m}^2/\text{s}}{\sqrt{\text{kg}^2\times\text{m}^2/\text{s}^2}} = \frac{\text{kg}\times\text{m}^2/\text{s}}{\text{kg}\times\text{m/s}} = \text{m}$$

41.6. **Solve:** **(a)**



(b) For n = 2, the probability of finding the particle at the center of the well is zero. This is because the wave function is zero at that point.
(c) This is consistent with standing waves. The n = 2 standing wave on a string has a node at the center of the string.

41.7. **Model:** The wave function decreases exponentially in the classically forbidden region.
Solve: The probability of finding a particle in the small interval δx at position x is Prob(in δx at x) = $|\psi(x)|^2\ \delta x$. Thus the ratio

$$\frac{\text{Prob(in } \delta x \text{ at } x = L+\eta)}{\text{Prob(in } \delta x \text{ at } x = L)} = \frac{|\psi(L+\eta)|^2\ \delta x}{|\psi(L)|^2\ \delta x} = \frac{|\psi(L+\eta)|^2}{|\psi(L)|^2}$$

The wave function in the classically forbidden region $x \geq L$ is

$$\psi(x) = \psi_{\text{edge}} e^{-(x-L)/\eta}$$

At the edge of the forbidden region, at $x = L$, $\psi(L) = \psi_{\text{edge}}$. At $x = L + \eta$, $\psi(L+\eta) = \psi_{\text{edge}}e^{-1}$. Thus

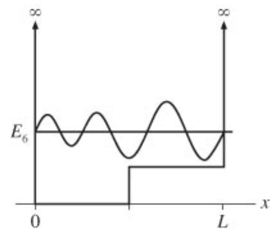
$$\frac{\text{Prob(in } \delta x \text{ at } x = L+\eta)}{\text{Prob(in } \delta x \text{ at } x = L)} = \frac{|\psi(L+\eta)|^2}{|\psi(L)|^2} = \frac{(\psi_{\text{edge}}e^{-1})^2}{(\psi_{\text{edge}})^2} = e^{-2} = 0.135$$

41.9. **Solve:** According to Equation 41.41, the penetration depth is $\eta = \hbar\big/\sqrt{2m(U_0 - E)}$. Hence,

$$U_0 - E = \frac{\hbar^2}{2m\eta^2} = \frac{\left(1.05\times10^{-34}\text{ J s}\right)^2}{2\left(9.11\times10^{-31}\text{ kg}\right)\left(1.0\times10^{-9}\text{ m}\right)^2} = 6.05\times10^{-21}\text{ J}\times\frac{1\text{ eV}}{1.6\times10^{-19}\text{ J}} = 0.038\text{ eV}$$

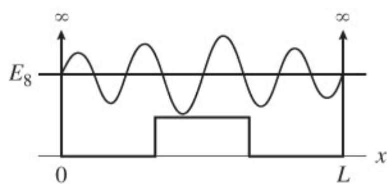
The electron's energy is 0.038 eV below U_0.

41.11. Visualize:



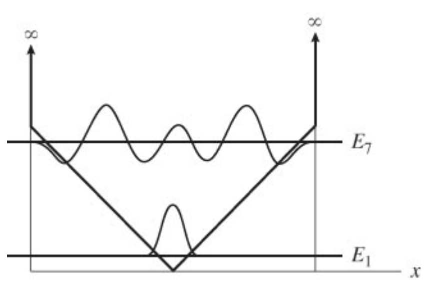
Solve: There are three factors to consider. First, the de Broglie wavelength increases as the particle's speed and kinetic energy decreases. Thus, the spacing between the nodes of $\psi(x)$ increases in regions where U is larger. Second, a particle is more likely to be found where it is moving the slowest. Thus, the amplitude of $\psi(x)$ increases in regions where U is larger. Third, for $n = 6$ there will be six antinodes to place.

41.12. Visualize:



Solve: There are three factors to consider. First, the de Broglie wavelength increases as the particle's speed and kinetic energy decreases. Thus, the spacing between the nodes of $\psi(x)$ increases in regions where U is larger. Second, a particle is more likely to be found where it is moving the slowest. Thus, the amplitude of $\psi(x)$ increases in regions where U is larger. Third, for $n = 8$ there will be eight antinodes to place.

41.14. Visualize:



The steps of Tactics Box 41.1 have been followed to sketch the wave functions shown in the figure.

41.23. Solve: A function $\psi(x)$ is a solution to the Schrödinger equation if

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi(x)$$

Let $\psi(x) = A\psi_1(x) + B\psi_2(x)$, where $\psi_1(x)$ and $\psi_2(x)$ are both known to be solutions of the Schrödinger equation. The second derivative of $\psi(x)$ is

$$\frac{d^2\psi}{dx^2} = \frac{d^2}{dx^2}\big(A\psi_1(x) + B\psi_2(x)\big) = A\frac{d^2\psi_1}{dx^2} + B\frac{d^2\psi_2}{dx^2}$$

Since $\psi_1(x)$ and $\psi_2(x)$ are solutions, it must be the case that

$$\frac{d^2\psi_1}{dx^2} = -\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi_1(x) \quad \text{and} \quad \frac{d^2\psi_2}{dx^2} = -\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi_2(x)$$

Using these results, the second derivative of $\psi(x)$ becomes

$$\frac{d^2\psi}{dx^2} = A\frac{d^2\psi_1}{dx^2} + B\frac{d^2\psi_2}{dx^2} = A\left(-\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi_1(x)\right) + B\left(-\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi_2(x)\right)$$

$$= -\frac{2m}{\hbar^2}\left[E - U(x)\right]\big(A\psi_1(x) + B\psi_2(x)\big)$$

$$= -\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi(x)$$

Thus $\psi(x)$ is a solution to the Schrödinger equation.

41.25. Model: Model the particle as a particle in a rigid one-dimensional box of length L.
Solve: **(a)** From Equation 41.22, the particle's energies are

$$E_n = \frac{n^2h^2}{8mL^2} \Rightarrow E_2 - E_1 = \frac{h^2}{8mL^2}\left(2^2 - 1^2\right) = \frac{3h^2}{8mL^2}$$

Since $E_2 - E_1 = hf = hc/\lambda_{2\to1}$, we have $\lambda_{2\to1} = 8mcL^2/3h$.
(b) The length of the box is

$$L = \sqrt{\frac{3h\lambda_{2\to1}}{8mc}} = \sqrt{\frac{3\left(6.63\times10^{-34}\text{ J s}\right)\left(694\times10^{-9}\text{ m}\right)}{8\left(9.11\times10^{-31}\text{ kg}\right)\left(3.0\times10^{8}\text{ m/s}\right)}} = 0.795\text{ nm}$$

41.27. Solve: From Equation 41.20, the wave functions for a particle in a box of length L are $\psi_n = A_n\sin\left(n\pi x/L\right)$. The wave function is nonzero only for $0 \le x < L$. The normalization requirement is

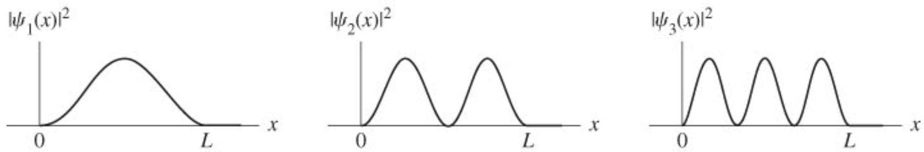
$$\int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx = A_n^2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = 1$$

Change the variable to $u = n\pi x/L$. Then, $dx = (L/n\pi)du$. The integration limits become $u = 0$ at $x = 0$ and $u = n\pi$ at $x = L$. The normalization integral, with the use of $\sin^2 u = \left(\frac{1}{2}\right)\left(1 - \cos 2u\right)$, becomes

$$1 = A_n^2\frac{L}{n\pi}\int_0^{n\pi}\sin^2 u\, du = A_n^2\frac{L}{n\pi}\int_0^{n\pi}\tfrac{1}{2}\left(1 - \cos 2u\right)du = A_n^2\frac{L}{n\pi}\left[\tfrac{1}{2}u - \tfrac{1}{4}\sin 2u\right]_0^{n\pi} = \frac{LA_n^2}{2} \Rightarrow A_n = \sqrt{\frac{2}{L}}$$

41.29. Model: Model the particle as being confined in a rigid one-dimensional box of length L.

Visualize:



Solve: (a) The probability density is $\left|\psi_n(x)\right|^2 = (2/L)\sin^2(n\pi x/L)$. Graphs of $\left|\psi_1(x)\right|^2$, $\left|\psi_2(x)\right|^2$, and $\left|\psi_2(x)\right|^2$ are shown above.

(b) The particle is most likely to be found at x where $\left|\psi(x)\right|^2$ is a maximum. See table in part (d).

(c) The particle is least likely to be found at x where $\left|\psi(x)\right|^2 = 0$. See table in part (d).

(d) The probability of finding the particle in the left one-third of the box is the area under the $\left|\psi(x)\right|^2$ curve between $x = 0$ and $x = \frac{1}{3}L$. From examining the graphs, we can determine whether this is more than, less than, or equal to one-third of the total area. The results are shown in the table below.

n	Most likely	Least likely	Probability in left one-third
1	$\frac{1}{2}L$	0 and L	$< \frac{1}{3}$
2	$\frac{1}{4}L$ and $\frac{3}{4}L$	$0,\ \frac{1}{2}L,$ and L	$> \frac{1}{3}$
3	$\frac{1}{6}L,\ \frac{3}{6}L,$ and $\frac{5}{6}L$	$0,\ \frac{1}{3}L,\ \frac{2}{3}L,$ and L	$= \frac{1}{3}$

(e) The probability of finding the particle in the range $0 \le x \le \frac{1}{3}L$ is

$$\text{Prob}\left(0 \le x \le \tfrac{1}{3}L\right) = \int_0^{L/3} \left|\psi_n(x)\right|^2 dx = \frac{2}{L}\int_0^{L/3} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

Change the variable to $u = n\pi x/L$. Then, $dx = (L/n\pi)du$. The integration limits become $u = 0$ at $x = 0$ m and $u = n\pi/3$ at $x = \frac{1}{3}L$. Then,

$$\text{Prob}\left(0 \le x \le \tfrac{1}{3}L\right) = \frac{2}{n\pi}\int_0^{n\pi/3} \sin^2 u\,du = \frac{2}{n\pi}\left[\tfrac{1}{2}u - \tfrac{1}{4}\sin 2u\right]_0^{n\pi/3} = \frac{1}{3} - \frac{1}{2n\pi}\sin\left(\frac{2n\pi}{3}\right)$$

The probability is 0.195 for $n = 1$, 0.402 for $n = 2$, and 0.333 for $n = 3$.

Assess: The results agree with the earlier estimates of the probability.