Physics 263: Electromagnetism and Modern Physics

Sections 0101- 0105

Exam 2

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Instructions:

- This is a closed book, closed notes exam to be completed in 50 minutes. You may use a basic scientific calculator, but no other aids are permitted. The final page provides a brief compilation of *possibly useful information*, including relevant physical constants.
- Work each problem in the space provided. If additional space is needed, use the back of the *previous* page and indicate that you have done so.
- Please write your name on each page, including this one. Do not use red ink.
- **Choose 4 out of 5 problems.** Use an X in the table below to indicate which problem to omit. If there is no X, we will grade the first four.
- **Explain your reasoning and show your work**. Partial credit will be awarded when the relevant physical principles are applied even if mistakes are made in execution of the steps. However, correct guesses without any explanation may be penalized.
- **Algebraic answers must have consistent dimensions and numerical answers must have consistent units**.
- **Honor pledge:** please copy and sign "I pledge on my honor that I have not given or received any unauthorized assistance on this examination."

Pledge:

Signature: ____________________ **Student ID number:** ________________

Printed name: Solutions **Section:** ________________

Problem	Score
1	
2	
3	
4	
5	
total	

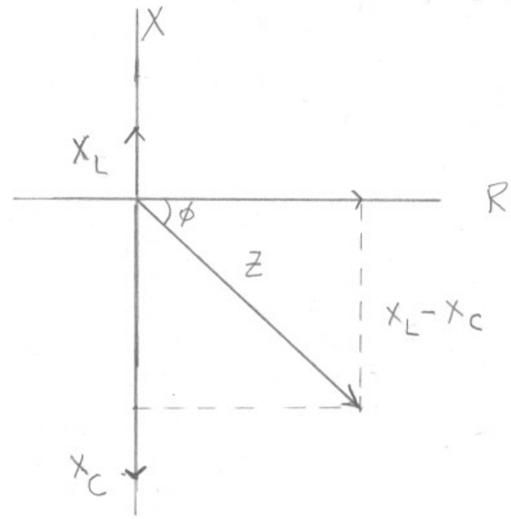
1. A series *RLC* circuit contains $R = 100\,\Omega$, L=200 mH, and C=50 μF.

a) Compute the impedances for each element at $\omega = 160$ radians/s and the net impedance. Draw a phasor diagram (to scale) that shows the net impedance and each of its contributions.*(10 pts)*

$X_C = \frac{1}{\omega C} = 125\,\Omega$

$X_L = \omega L = 32\,\Omega$

$Z = \left[R^2 + (X_c - X_L)^2\right]^{1/2} = 136.6\,\Omega$



b) Compute the phase shift between the voltage and the current. Does the current lead or lag the voltage? Justify your answer using the phasor diagram. *(10 pts)*

$\tan\phi = \frac{X_L - X_C}{R} \Rightarrow \phi = -42.9^0$ with current leading voltage.

Recall: $V = IZ$, $\Delta V_R = IR$ with $I(t) = I_{max} \sin\omega t$. Hence, the current is in phase with ΔV_R and leads the source voltage because $X_c > X_L \Rightarrow \phi < 0$.

c) Compute the resonant frequency for this circuit. *(5 pts)*

$X_L = X_C \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} = 316.2$ rad/s

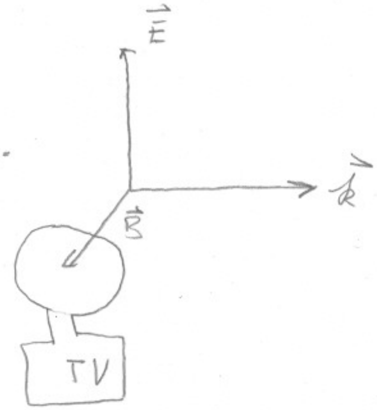
$f_0 = \frac{\omega_0}{2\pi} = 50.3$ Hz

Name: ______________________________

2. A UHF antenna consists of a single circular loop of radius *r*. The voltage in the antenna is produced by the changing magnetic flux.

a) Sketch the orientation of the antenna relative to the directions of the electric and magnetic fields and the propagation of the electromagnetic signal for the conditions that give the strongest signal in the antenna. Explain. *(10 pts)*

To obtain maximum emf in the UHF antenna, we need $\vec{B}$ normal to the plane of the loop to maximize Φ_B. $\vec{k}$ is in the direction from transmitter to receiver and lies in the plane of the loop. $\vec{E}$ is $\perp$ to $\vec{k}$ and $\vec{B}$.



b) Suppose that the antenna has a radius of 10 cm and is located 100 km from a station that broadcasts isotropically (equally in all directions) a total average power of 750 kW using a frequency of 600 MHz. Compute the rms (root mean square) voltage induced in the antenna when it is oriented for best reception. *(5 pts)*

$$\mathcal{E} = -\dot{\Phi}_B, \quad \Phi_B = \pi r^2 B_0 \sin\omega t \Rightarrow \mathcal{E} = -\omega B_0 \pi r^2 \cos\omega t$$

$$\langle \mathcal{E}^2 \rangle = \tfrac{1}{2}(\omega B_0 \pi r^2)^2 \Rightarrow \mathcal{E}_{rms} = \pi r^2 \omega B_{rms}, \quad B_{rms} = \sqrt{\tfrac{1}{2}}\, B_0$$

$$I = \frac{P}{4\pi d^2} = \frac{c B_{rms}^2}{\mu_0} \Rightarrow B_{rms} = \left(\frac{\mu_0}{4\pi}\frac{P}{cd^2}\right)^{1/2} = \left(10^{-7}\frac{N}{A^2}\,\frac{750\,kW}{3\times10^8\frac{m}{s}(100\,km)^2}\right)^{1/2}$$

$$B_{rms} = 1.58\times10^{-10}\ T$$

$$\mathcal{E}_{rms} = 18.7\ mV$$

c) Suppose that the antenna has inductance *L* and is one element of a series *RLC* circuit with fixed resistance *R* and variable capacitance *C*. How should *C* be adjusted to receive frequency *f*? How does *R* affect the quality of the tuner? *(10 pts)*

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi f \Rightarrow C = [(2\pi f)^2 L]^{-1}$$

R governs the quality factor, $\frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}$, for the tuner.

Small R $\Rightarrow$ strong discrimination between nearby channels.

3. An object is placed at distance 15 cm *to the right* of a thin converging lens with focal length 12 cm. Construct a ray diagram that includes at least two of the principal rays and determine the position, orientation, and magnification (or reduction) of the image. Is the image real or virtual? *(25 pts)*

Note: although you may check your answers using the thin-lens equation, to obtain full credit you must produce a neat ray diagram and derive your answers from it geometrically.

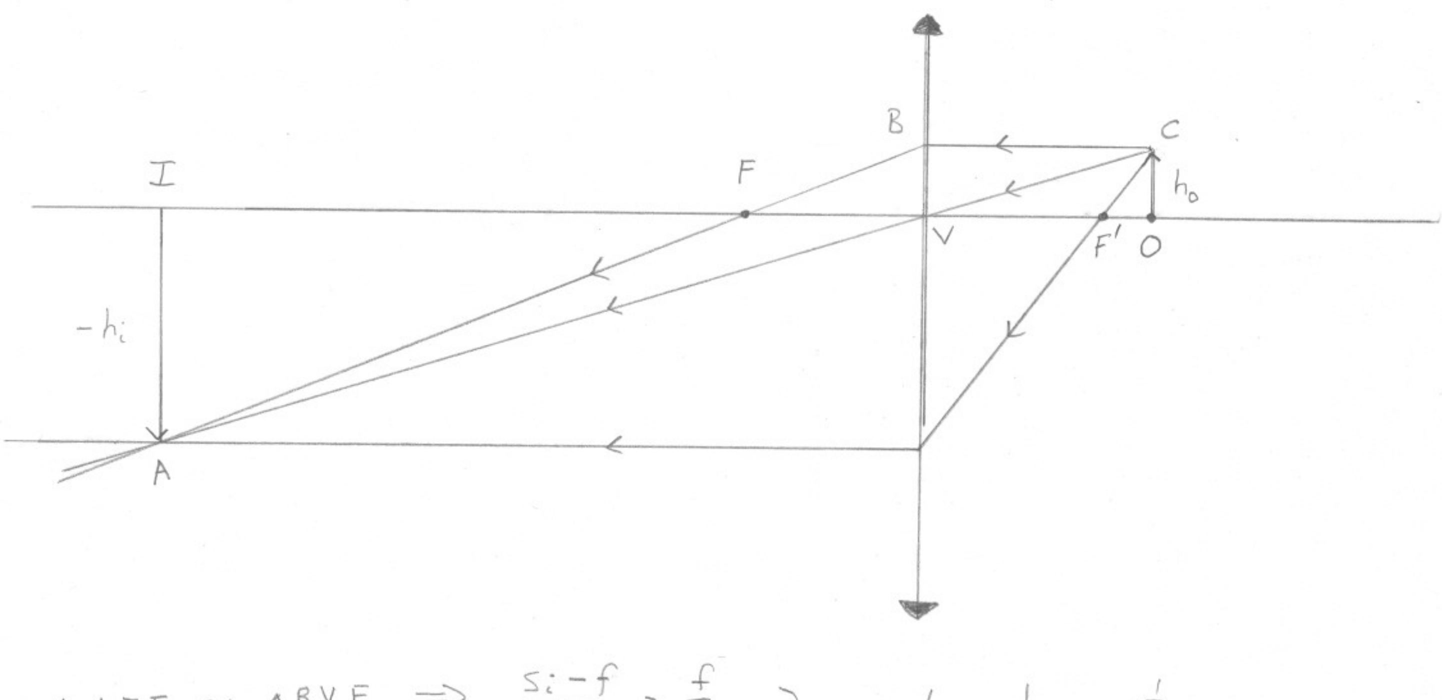
Image type: real or virtual? (circle one)

Image position = 60 cm to left of lens

Image orientation: erect or inverted? (circle one)

Magnification factor = -4

Diagram:



$\triangle AIF \sim \triangle BVF \Rightarrow \dfrac{s_i - f}{-h_i} = \dfrac{f}{h_o}$

$\triangle AVI \sim \triangle CVB \Rightarrow \dfrac{s_i}{-h_i} = \dfrac{s_o}{h_o}$

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

$$M = -\frac{s_i}{s_o}$$

$f = 12\ \text{cm},\ s_o = 15\ \text{cm} \Rightarrow s_i = 60\ \text{cm} \quad M = -4$

Although the diagram is not perfect, these dimensions are roughly to scale.

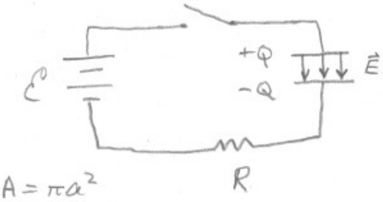
4. A series *RC* circuit consists of a switch that is initially open, a battery with potential $\mathcal{E}$, a resistor *R*, and a parallel-plate capacitor that has circular plates of radius a separated by distance ℓ. Assume that $a >> \ell$ such that the electric field within the capacitor is nearly uniform. The switch is closed at $t = 0$.

a) Determine the time dependence of the electric field within the capacitor. *(6 pts)*

$$\mathcal{E} = \frac{Q}{C} + \dot{Q}R \quad , \quad Q(t=0) = 0$$

$$Q(t) = \mathcal{E}C\left(1 - e^{-t/\tau}\right) \quad \text{with } \tau = RC$$

$$\text{check: } \dot{Q} = \frac{\mathcal{E}C}{\tau}e^{-t/\tau} = \frac{\mathcal{E}}{R}e^{-t/\tau} \Rightarrow \dot{Q}R + \frac{Q}{C} = \mathcal{E} \checkmark$$

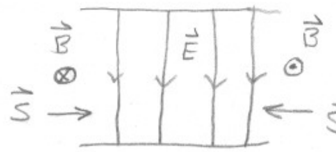


$$\text{Gauss} \Rightarrow E = \frac{Q}{\varepsilon_0 A} = \frac{Q}{C\ell} \Rightarrow C = \frac{\varepsilon_0 A}{\ell} \quad \text{with } A = \pi a^2$$

$$E = \frac{\mathcal{E}}{\ell}\left(1 - e^{-t/\tau}\right)$$

b) Determine the magnetic field as a function of the distance *r* from axis of the capacitor. *(6 pts)*

Apply Ampere-Maxwell law to circle of radius r centered on axis



$$\le a \Rightarrow \quad 2\pi r B = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \pi r^2 \frac{\dot{Q}}{\varepsilon_0 A}$$

$$\therefore B = \frac{\mu_0 r}{2\pi a^2}\frac{\mathcal{E}}{R}e^{-t/\tau}$$

c) Compute the electromagnetic energy flux at the cylindrical surface of the capacitor. What is its direction? *(6 pts)*

$\vec{S}$ is radially inward as shown above.

$$S = \frac{EB}{\mu_0} = \frac{\mathcal{E}^2}{2\pi a \ell R}e^{-t/\tau}\left(1 - e^{-t/\tau}\right) \quad \text{at } r = a$$

d) Compare the total energy that enters the capacitor with the stored energy as $t \to \infty$. *(7 pts)*

$$U = \int_0^\infty dt\, S(t)\, 2\pi a \ell = \frac{\mathcal{E}^2}{R}\int_0^\infty dt\, e^{-t/\tau}\left(1 - e^{-t/\tau}\right) = \frac{\mathcal{E}^2}{R}\frac{\tau}{2}$$

$$\therefore U = \frac{1}{2}C\mathcal{E}^2 \quad \text{as expected for capacitor.}$$

This can be compared to the energy in the electric field:

$$U_E = \frac{\varepsilon_0 E^2}{2}\pi a^2 \ell = \frac{\varepsilon_0 \mathcal{E}^2}{2\ell^2}\pi a^2 \ell = \frac{1}{2}C\mathcal{E}^2 \quad \text{with } C = \frac{\varepsilon_0}{\ell}\pi a^2$$

5. The apparent position of the sun is affected by refraction in the earth's atmosphere. Suppose that we represent the earth as a sphere of radius R surrounded by a thin atmospheric shell of thickness h with uniform refractive index n. The local horizontal plane is tangent to the observer's position on the surface.

a) At sunset the sun *appears* to be in the horizontal plane. Is its actual position above or below the horizontal plane? Explain. *(5 pts)*

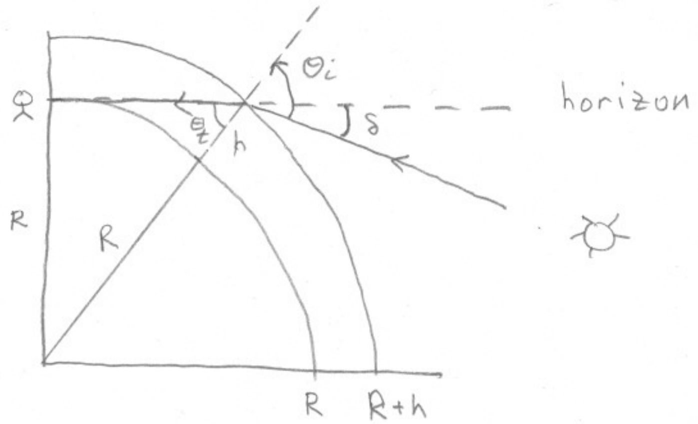
The sun's rays are refracted toward the normal which, as shown below, is radial at the point of entry into the atmospheric shell. Therefore, at sunset the sun is actually below the geometrical horizon.

b) Find an expression for the angle δ that the sun's rays outside the earth's atmosphere make with respect to the horizontal plane at sunset. *(10 pts)*

$$\sin\theta_i = \sin(\theta_t + \delta) = n \sin\theta_t$$

$$\sin\theta_t = \frac{R}{R+h}$$

$$\therefore\ \delta = \sin^{-1}\left(\frac{nR}{R+h}\right) - \sin^{-1}\left(\frac{R}{R+h}\right)$$



c) Find a simple approximation for δ assuming that $h << R$, $0 < n-1 << 1$. *(5 pts)*

$$\delta << \theta_t \Rightarrow \left.\begin{array}{l} \sin(\theta_t + \delta) \approx \sin\theta_t + \delta\cos\theta_t \\ \sin(\theta_t + \delta) = n\sin\theta_t \end{array}\right\} \quad \delta = (n-1)\tan\theta_t$$

$$h << R \Rightarrow \quad \sin\theta_t \approx 1 - \frac{h}{R}, \quad \cos\theta_t \approx \sqrt{\frac{2h}{R}} \Rightarrow \tan\theta_t \approx \sqrt{\frac{R}{2h}}$$

$$\therefore\ \delta \approx (n-1)\sqrt{\frac{R}{2h}}$$

d) Compute δ using $n-1 = 3\times10^{-4}$, $h = 20$ km, $R = 6.4\times10^{6}$ m. Compare your result with the angular diameter of the sun, approximately half a degree. *(5 pts)*

$\delta = 3.79$ mrad $= 0.22^\circ$ $\approx$ half the angular diameter of the s

Possibly useful information

$\oint d\vec{a}\cdot\vec{E} = \dfrac{Q}{\varepsilon_0}$

$\oint d\vec{a}\cdot\vec{B} = 0$

$\oint d\vec{s}\cdot\vec{E} = -\dfrac{d\Phi_B}{dt}$

$\oint d\vec{s}\cdot\vec{B} = \mu_0\left(I + \varepsilon_0\dfrac{d\Phi_E}{dt}\right)$

$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P}$

$\vec{B} = \mu\vec{H} = \mu_0(\vec{H} + \vec{M})$

static: $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\int dq\dfrac{\hat{r}}{r^2}$

static: $\vec{B} = \dfrac{\mu_0}{4\pi}\int\dfrac{I\,d\vec{s}\times\hat{r}}{r^2}$

$u = \dfrac{\varepsilon_0 E^2}{2} + \dfrac{B^2}{2\mu_0}$

$\vec{S} = \vec{E}\times\vec{B}/\mu_0$

$\dfrac{1}{f} = (n-1)(\dfrac{1}{R_1} - \dfrac{1}{R_2})$

$\dfrac{1}{s_o} + \dfrac{1}{s_i} = \dfrac{1}{f}$

$c^2 = \dfrac{1}{\varepsilon_0\mu_0}$

$c = 2.9979\times10^8\ \dfrac{m}{s}$

$\varepsilon_0 = 8.8542\times10^{-12}\ \dfrac{F}{m}$

$\mu_0 = 4\pi\times10^{-7}\ \dfrac{N}{A^2}$

$e = 1.6022\times10^{-19}\ C$

$N_A = 6.022\times10^{23}\ mole^{-1}$

$m_e = 9.109\times10^{-31}\ kg = 0.511\,MeV/c^2$

$m_p = 1.672\times10^{-27}\ kg = 938.27\,MeV/c^2$

$\mu_B = 9.27\times10^{-24}\ \dfrac{J}{T}$

$\mu_N = 5.05\times10^{-27}\ \dfrac{J}{T}$

Dimensions and SI Units for Basic Electromagnetic Quantities

Quantity	**Unit**	**Abbreviation**	**Conversions**	**Dimensions**
Charge	Coulomb	C		Q
Electric potential	Volt	V	J/C	$ML^2T^{-2}Q^{-1}$
Electric field	Volt/meter	V/m	N/C	$MLT^{-2}Q^{-1}$
Capacitance	Farad	F	C/V	$M^{-1}L^{-2}T^2Q^2$
Current	Ampere	A	C/s	QT^{-1}
Magnetic field	Tesla	T	$N/(A\ m)$	$MT^{-1}Q^{-1}$
Magnetic flux	Weber	Wb	J/A	$ML^2T^{-1}Q^{-1}$
Inductance	Henry	H	J/A^2	ML^2Q^{-2}
Resistance	ohm	Ω	V/A	$ML^2T^{-1}Q^{-2}$