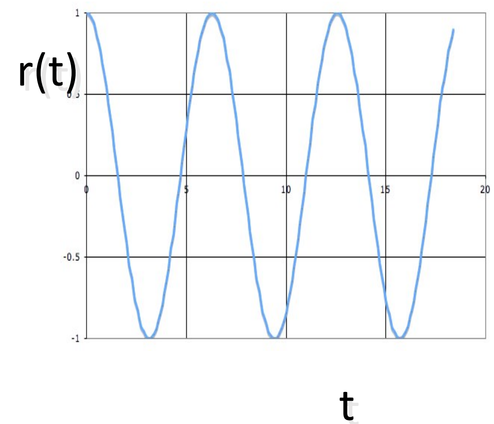
Oscillations and Waves

What's the difference?

Separation between two atoms in a molecule r(t)



Oscillations involve a discrete set of quantities that vary in time (usually periodically).

Examples: pendula, vibrations of individual molecules, firefly lights, currents in circuits.

Waves involve continuous quantities the vary in both space and time. (variation may be periodic or not)

Examples: light waves, sound waves, elastic waves, surface waves, electro-chemical waves on neurons

Learning about Oscillations and waves

- Why to learn it
  - How the ear senses sound
  - Sound itself
  - Brain waves
  - Heart contraction waves
  - Molecule oscillations
- What to learn
  - How to describe oscillations mathematically (sin, cos)
  - How to think about waves
  - Resonances

- Heart beat
- http://www.youtube.com/watch?annotation_id=annotation_611436&feature=iv&src_vid=Pes9O5z8efk&v=uR4t__B-Zwg

- Ventricular Fibrillation

http://www.youtube.com/watch?v=riUAFkV7HCU

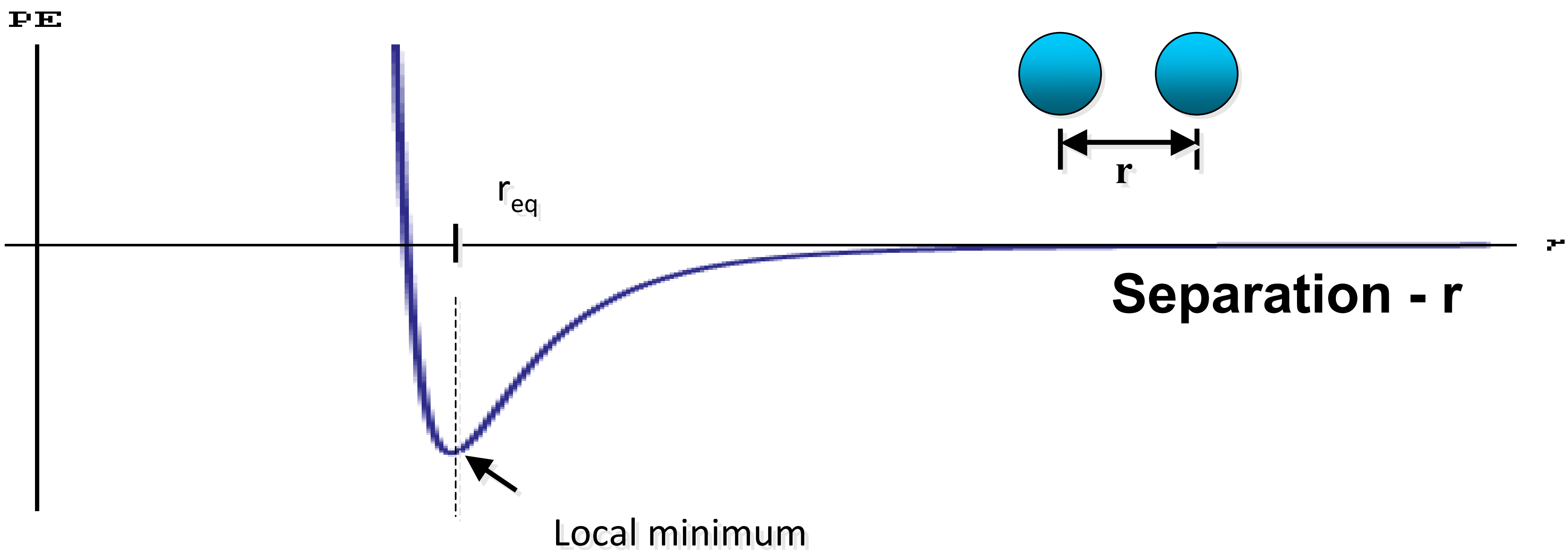
What are the key ingredients for oscillatory motion?

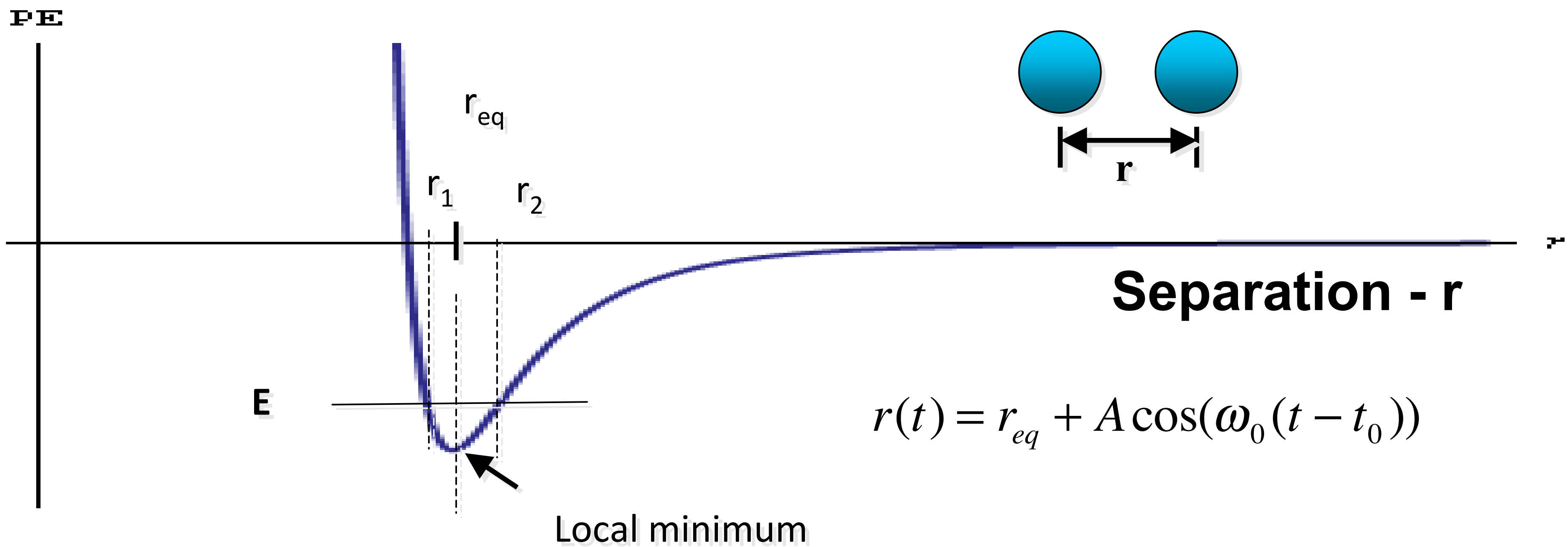
1. Motion given by Newton's laws a = F/m

2. Force F is derived from a potential with a local minimum.

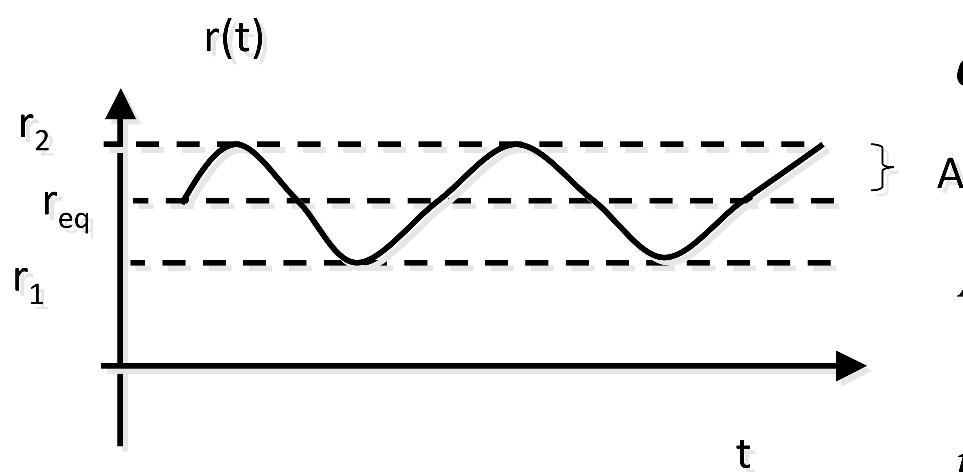
$$F(r) = -\frac{dU(r)}{dr} \simeq -k(r - r_{eq}) \qquad U(r) \simeq \frac{k}{2}(r - r_{eq})^2 + U(r_{eq})$$

Potential Energy - U (r)





$$r(t) = r_{eq} + A\cos(\omega_0(t - t_0))$$

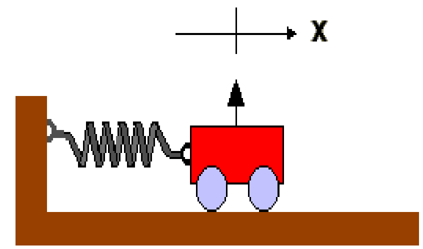


$$\omega_0 = \sqrt{\frac{k}{m}}$$ natural frequency of oscillation

$$A = (r_2 - r_{eq})$$ Amplitude of Oscillation

$$t_0 =$$ Any time $r(t)=r_2$

Model system: Mass on a Spring

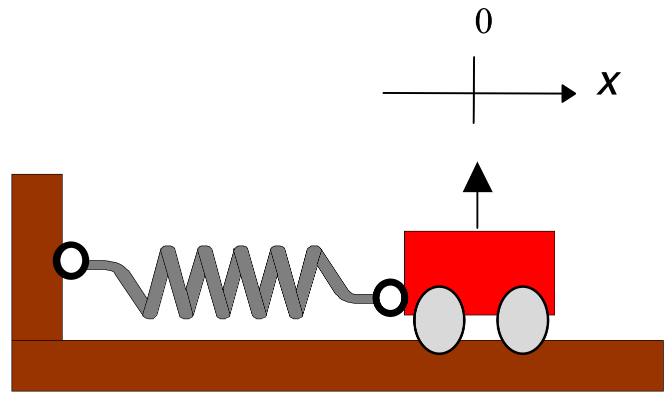


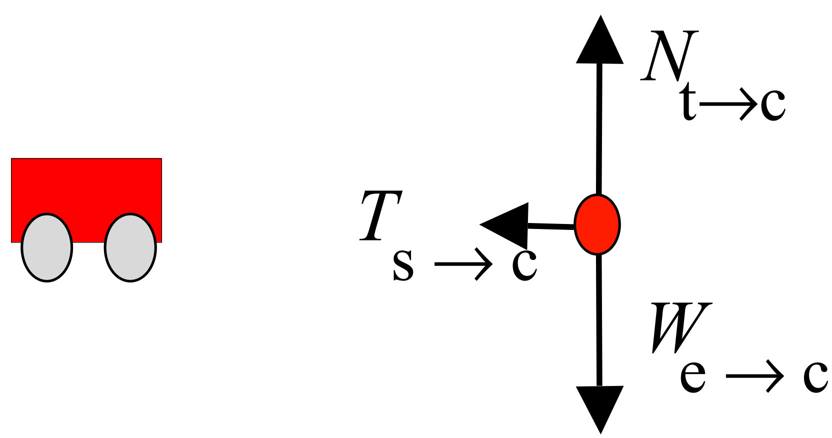
- Consider a cart of mass *m* attached to a light (mass of spring << *m*) spring.
- Choose the coordinate system so that when the cart is at 0 the spring it at its rest length
- Recall the properties of an ideal spring.
  - When it is pulled or pushed on both ends it changes its length.
  - T is tension, T>0 means the spring is being stretched

$$T = k\Delta l$$

Analyzing the forces: cart & spring

- What are the forces acting on the cart?



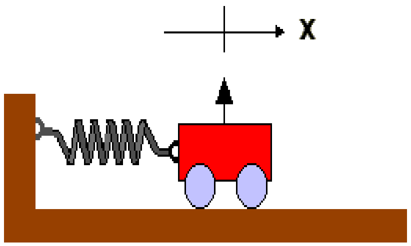


What do the subscripts mean ?

A mass connected to a spring is oscillating back and forth. Consider two possibilities:

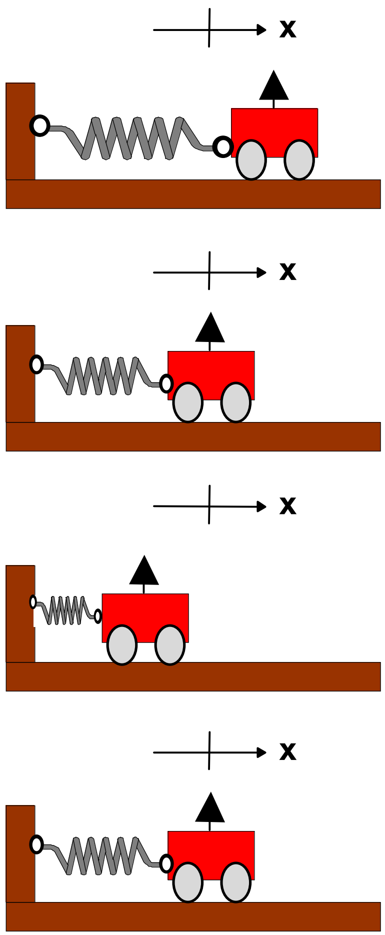
(i) at some point during the oscillation the mass has $v = 0$ but $a \neq 0$

(ii) at some point during the oscillation the mass has $v = 0$ and $a = 0$.



1. Both occur sometime during the oscillation.
2. Neither occurs during the oscillation.
3. Only (i) occurs.
4. Only (ii) occurs.

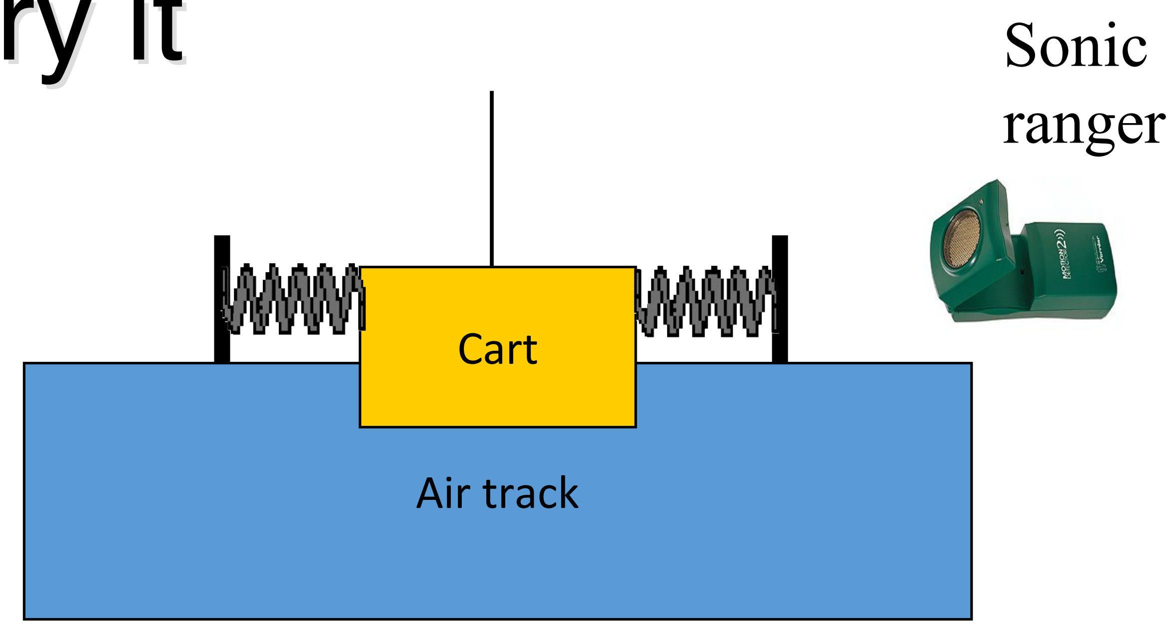
Tracking the motion



x	F^{net}	a	v
→	←	←	0

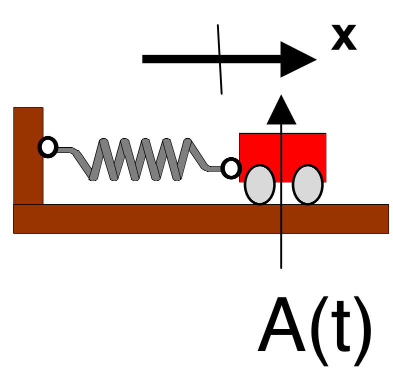
Whiteboard, TA & LA

Let's try it

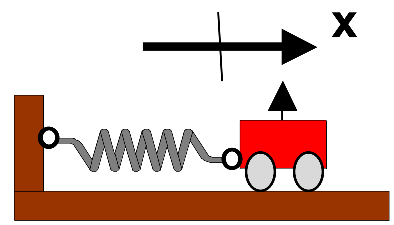


Why do we have two springs?

- Position of the cart depends on time t
- Lets call the x position of the cart: A(t)



Doing the Math: The Equation of Motion



- Newton's equation for the cart is

$$a = \frac{F_{net}}{m} = \frac{-k\,x(t)}{m} = -\left(\frac{k}{m}\right)x(t)$$

Acceleration is defined

$$a = \frac{d^2x(t)}{dt^2}$$

Solving a differential equation

- Express *acceleration a* as a derivative of x(t).

$$a = \frac{d^2x(t)}{dt^2} = -\left(\frac{k}{m}\right)x(t)$$

- Verify solution $x(t) = A\cos(\omega_0(t - t_0))$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

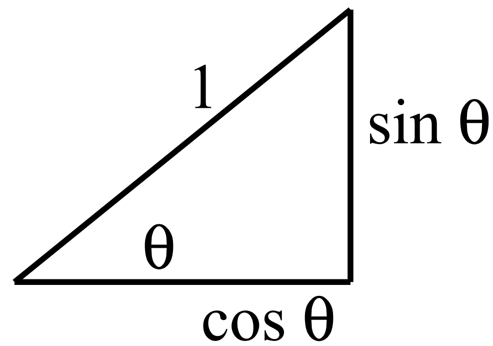
$$\frac{d^2\cos\theta}{d\theta^2} = -\cos\theta$$

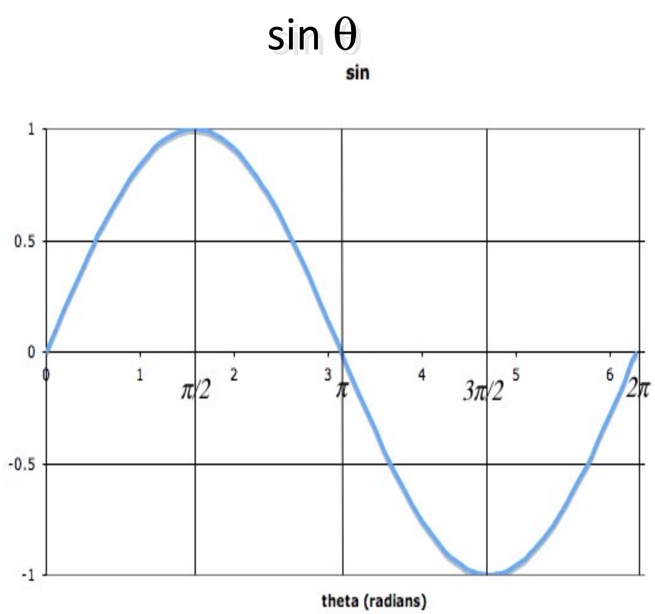
What determines A and t_0 ?

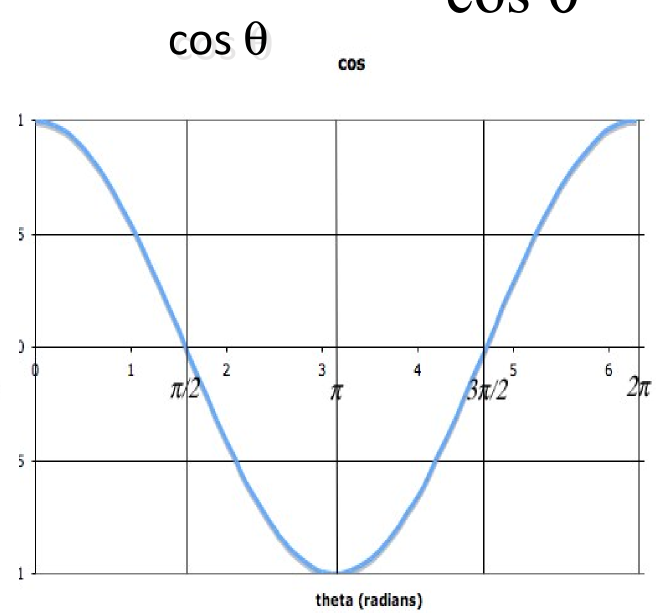
Whiteboard, TA & LA

Graphs: sin(θ) vs cos(θ)

- Which is which? How can you tell?
- The two functions sin and cos are derivatives of each other (slopes), but one has a minus sign.
  Which one?
  How can you tell?





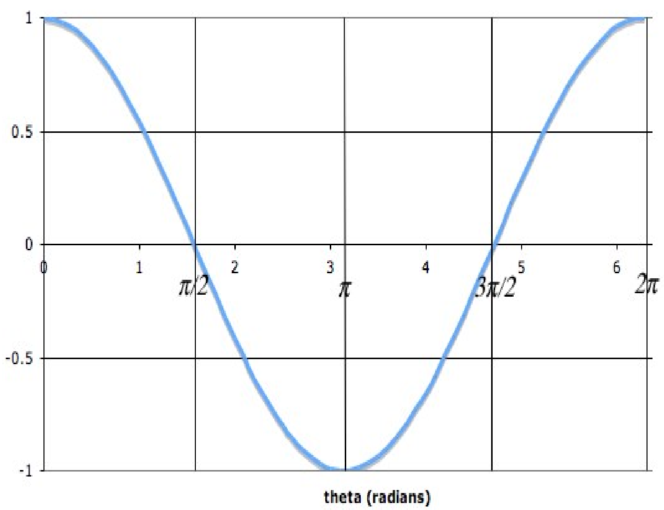


Graphs: $\sin(\theta)$ vs $\sin(\omega_0 t)$

- For angles, $\theta = 0$ and $\theta = 2\pi$ are the same
  so you only get one cycle.
- For time, t can go on forever so the cycles repeat.

What does changing ω_0 do to this graph?

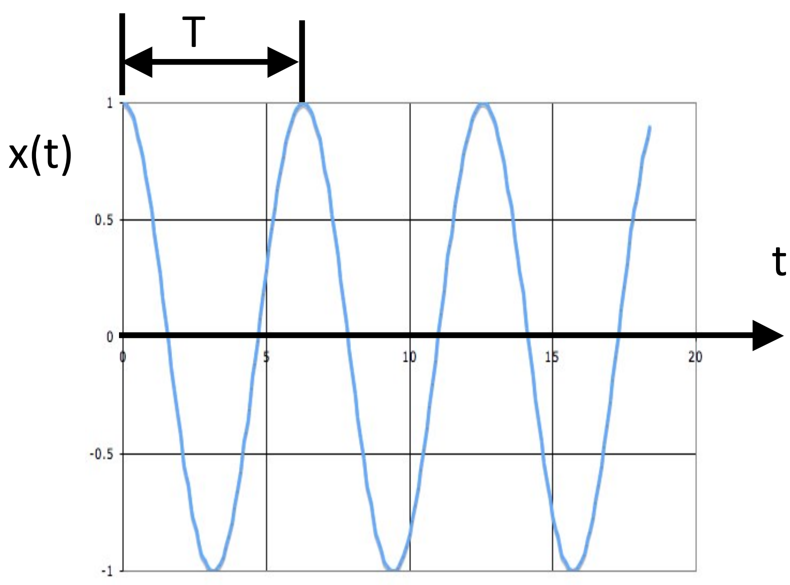
$\cos(\theta)$



$\cos(\omega_0 t)$

Interpreting the Result

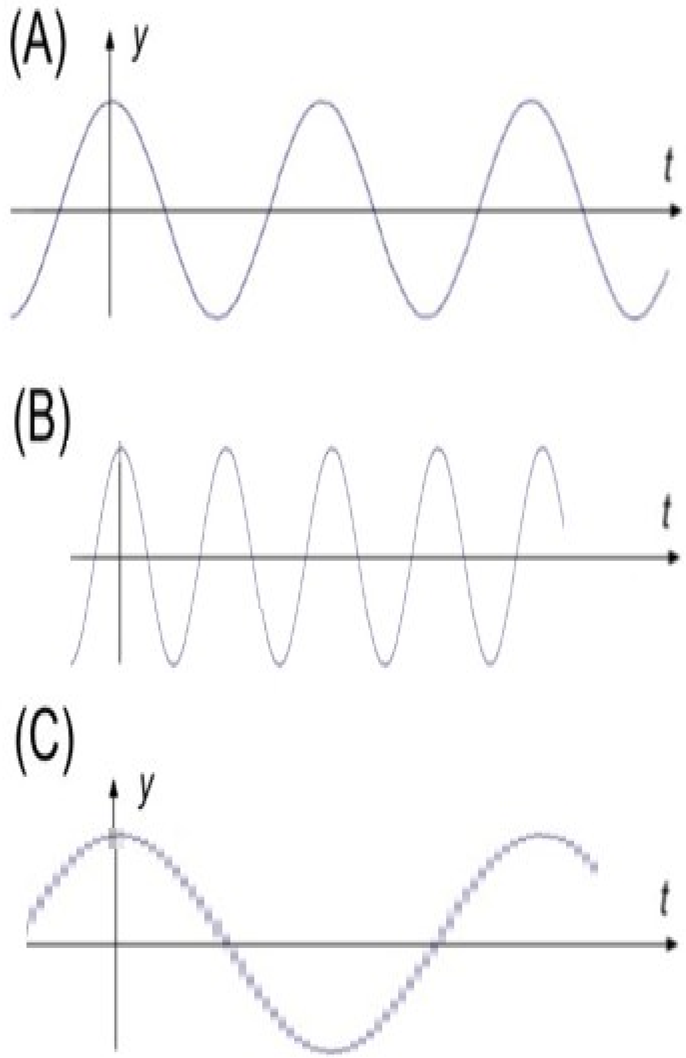
- What do the various terms mean?
  - *A* is the maximum displacement — the ***amplitude*** of the oscillation.
  - What is ω_0? If T is the ***period*** (how long it takes to go through a full oscillation) then



$$\omega_0 t : 0 \rightarrow 2\pi$$

$$t \quad : 0 \rightarrow T$$

$$\omega_0 T = 2\pi \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T}$$



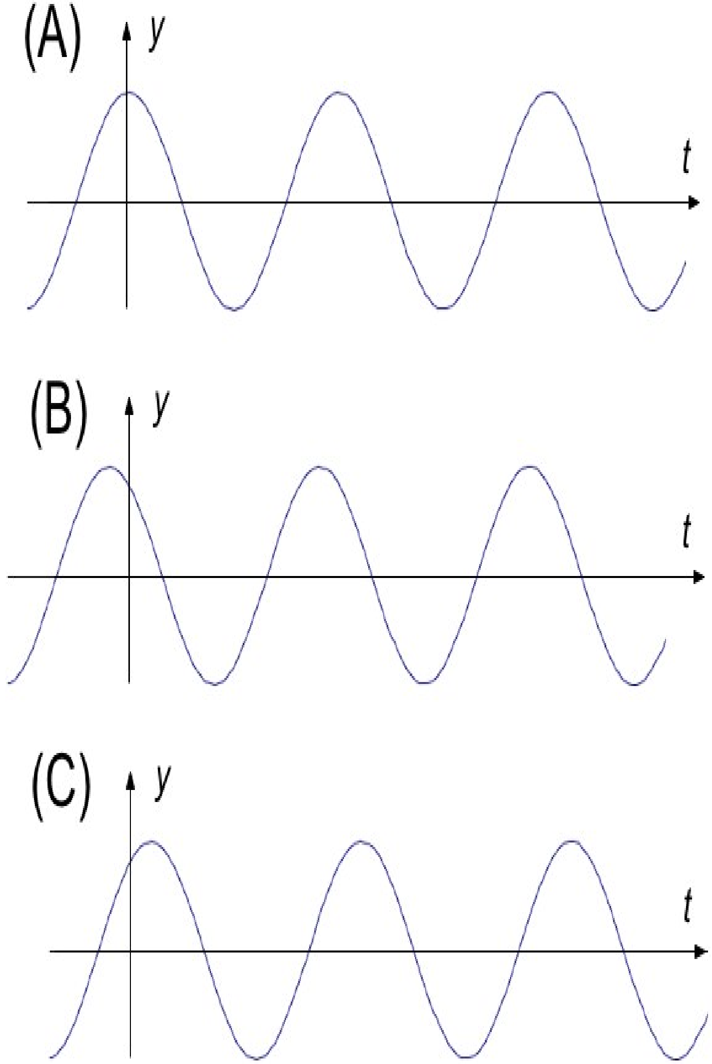
If curve (A) is

$$A\cos(\omega_0 t)$$

which curve is

$$A\cos(2\omega_0 t)?$$

1. (A)
2. (B)
3. (C)
4. None of the above.



Which of these curves is described by

$$A\cos(\omega_0 t + \phi)$$

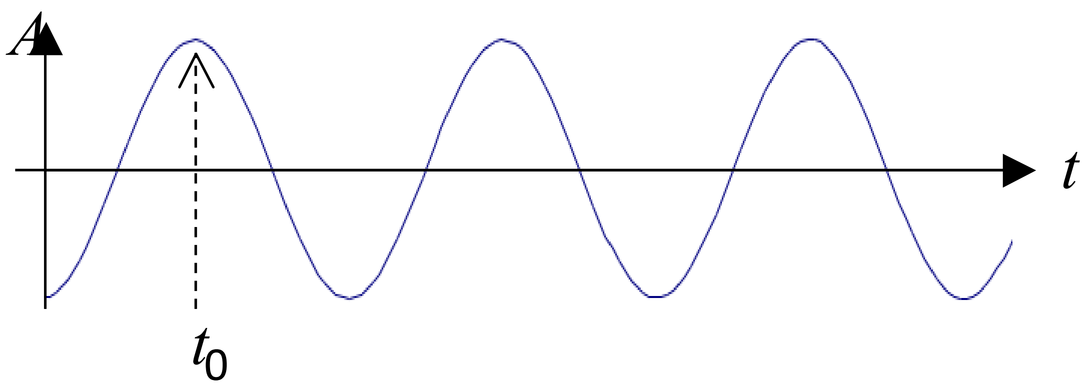
with $\phi > 0$ (and $\phi << 2\pi$)?

1. (A)
2. (B)
3. (C)
4. None of the above.

Whiteboard, TA & LA

Oscillations

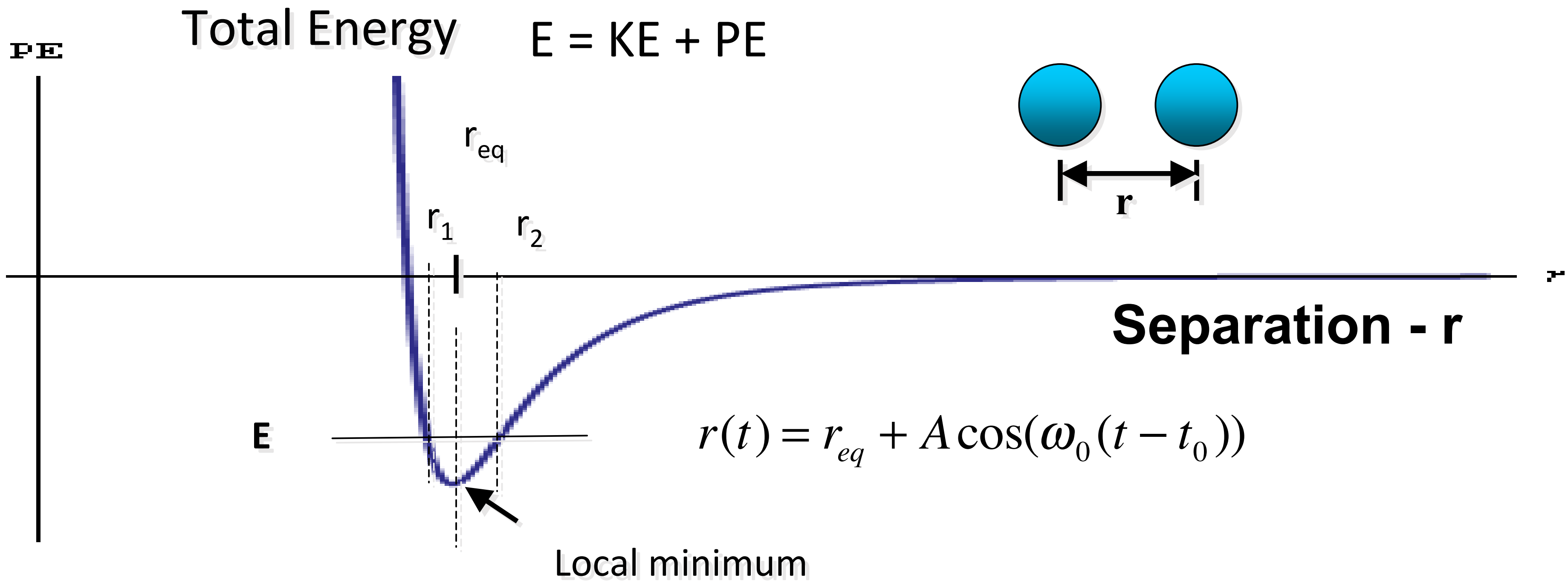
- A typical oscillation



$$x(t) = A\cos(\omega_0(t - t_0))$$

$$A(t) = A_0 \cos(\omega_0(t - t_0))$$

$$= A_0 \cos(\omega_0 t - \omega_0 t_0) = A_0 \cos(\omega_0 t - \phi)$$



E = KE + PE

$$r(t) = r_{eq} + A\cos(\omega_0(t - t_0))$$

Potential PE = U

$$U(r) \simeq \frac{k}{2}(r - r_{eq})^2 + U(r_{eq})$$

$$U = \frac{k}{2}A^2\cos^2(\omega(t - t_0)) + U(r_{eq})$$

Kinetic

$$KE = \frac{m}{2}(v(t))^2 = \frac{m}{2}(\frac{dr(t)}{dt})^2$$

$$KE = \frac{m\omega_0^2}{2}A^2\sin^2(\omega(t - t_0))$$

add them together

Whiteboard, TA & LA

$$\omega_0^2 = \frac{k}{m}$$

$$E = U(r_{eq}) + \frac{k}{2} A^2 \cos^2(\omega(t - t_0)) + \frac{m\omega_0^2}{2} A^2 \cos^2(\omega(t - t_0))$$

$$= U(r_{eq}) + \frac{k}{2} A^2 \left[\cos^2(\omega(t - t_0)) + \sin^2(\omega(t - t_0)) \right]$$

$$= U(r_{eq}) + \frac{k}{2} A$$

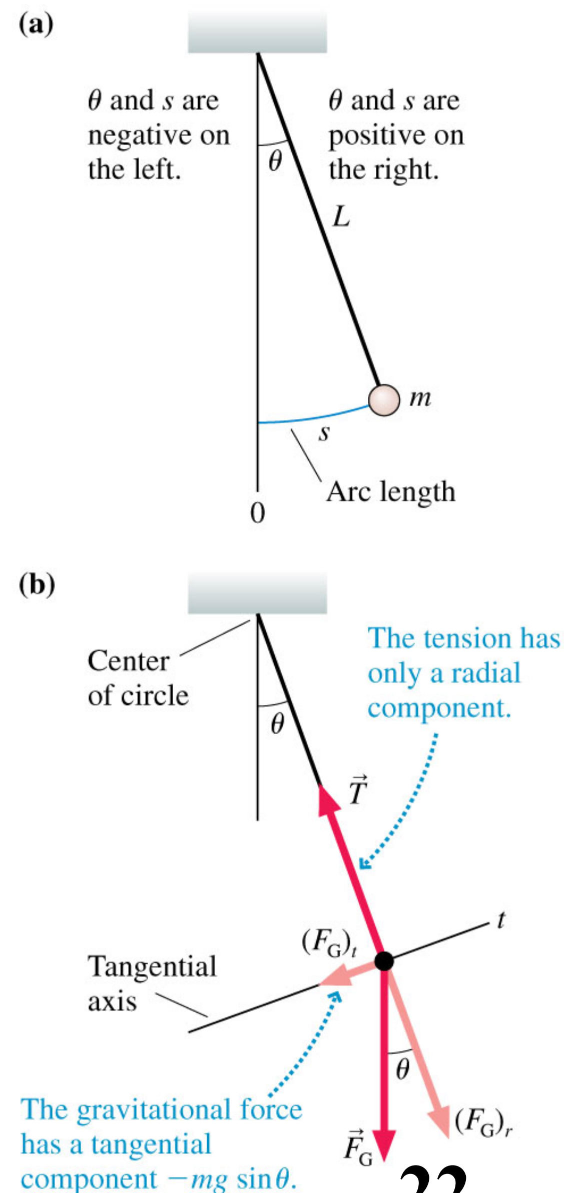
Total energy is constant

The Simple Pendulum

- Consider a mass m attached to a string of length L which is free to swing back and forth.
- If it is displaced from its lowest position by an angle θ, Newton's second law for the tangential component of gravity, parallel to the motion, is:

$$(F_{\text{net}})_t = \sum F_t = (F_G)_t = -mg\sin\theta = ma_t$$

$$\frac{d^2s}{dt^2} = -g\sin\theta$$



The Simple Pendulum

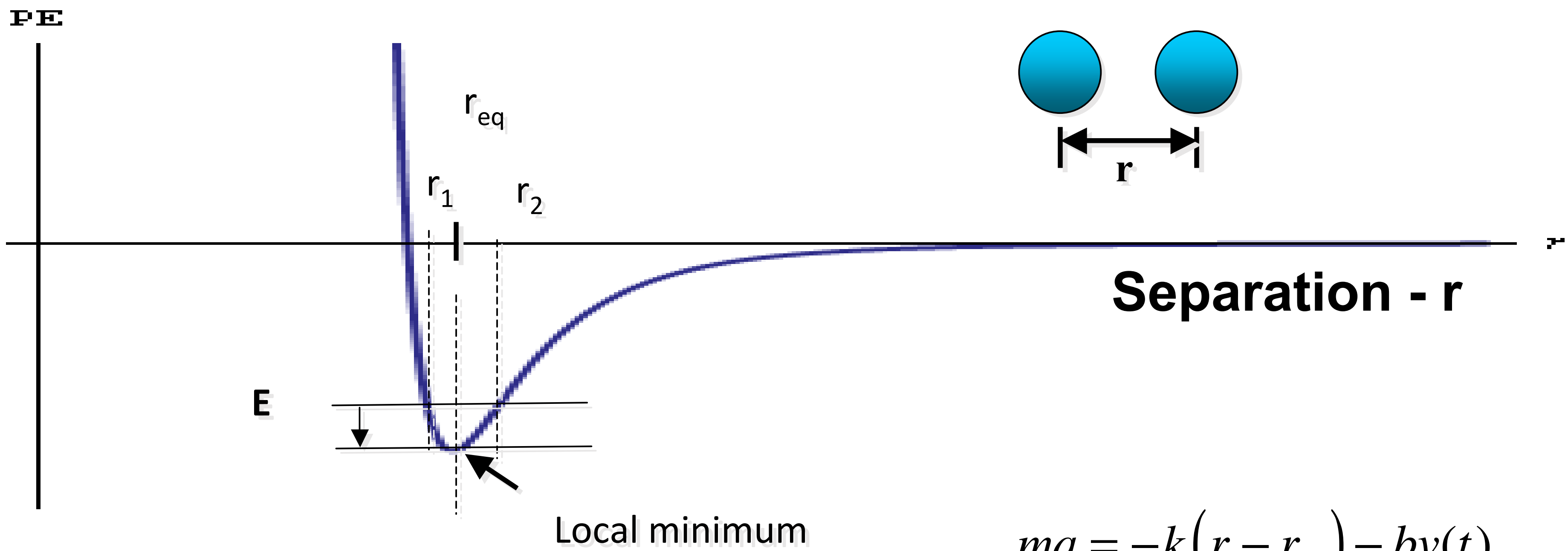
If we restrict the pendulum's oscillations to small angles (< 10°), then we may use the **small angle approximation** sin $\theta \approx \theta$, where θ is measured in radians.

$$(F_{\text{net}})_t = -mg\sin\theta \approx -mg\theta = -\frac{mg}{L}s$$

and the angular frequency of the motion is found to be:

$$\omega = 2\pi f = \sqrt{\frac{g}{L}}$$

Now add friction



r(t)

$$ma = -k\left(r - r_{eq}\right) - bv(t)$$

$$\frac{d^2}{dt^2} r(t) = -\omega_0^2 r(t) - \gamma \frac{dr}{dt}$$

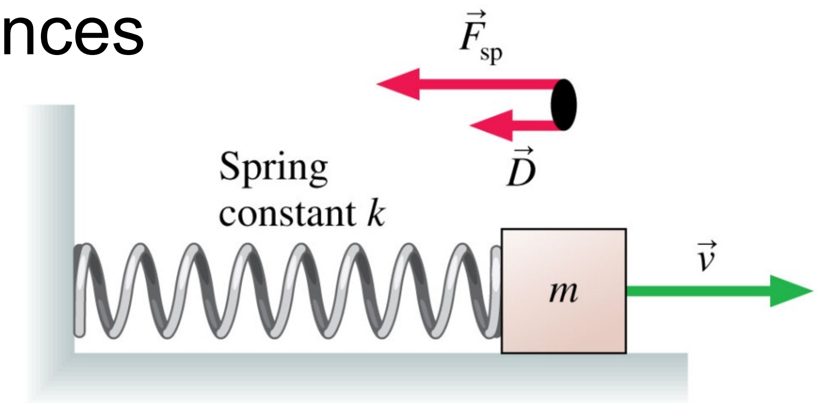
$$r(t) = r_{eq} + A\exp[-\frac{\gamma}{2}(t - t_0)]\cos(\omega_1(t - t_0))$$

$$\omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

$$\gamma = \frac{b}{m}$$

Damped Oscillations

When a mass on a spring experiences the force of the spring as given by Hooke's Law, as well as a linear drag force of magnitude $|D| = bv$, the solution is:



$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0) \qquad \text{(damped oscillator)}$$

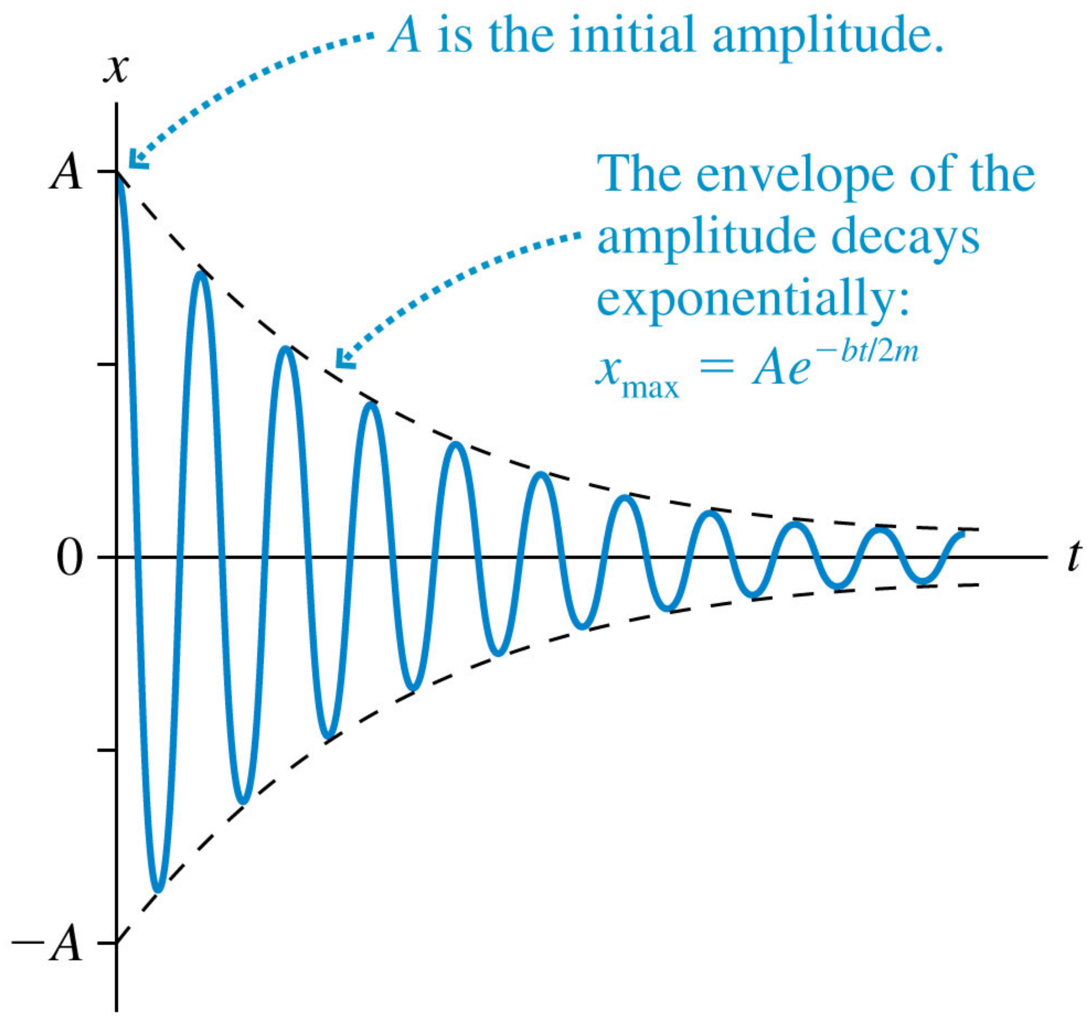
where the angular frequency is given by:

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

Here $\omega_0 = \sqrt{k/m}$ is the angular frequency of the undamped oscillator ($b = 0$).

Damped Oscillations

Position-versus-time graph for a damped oscillator.

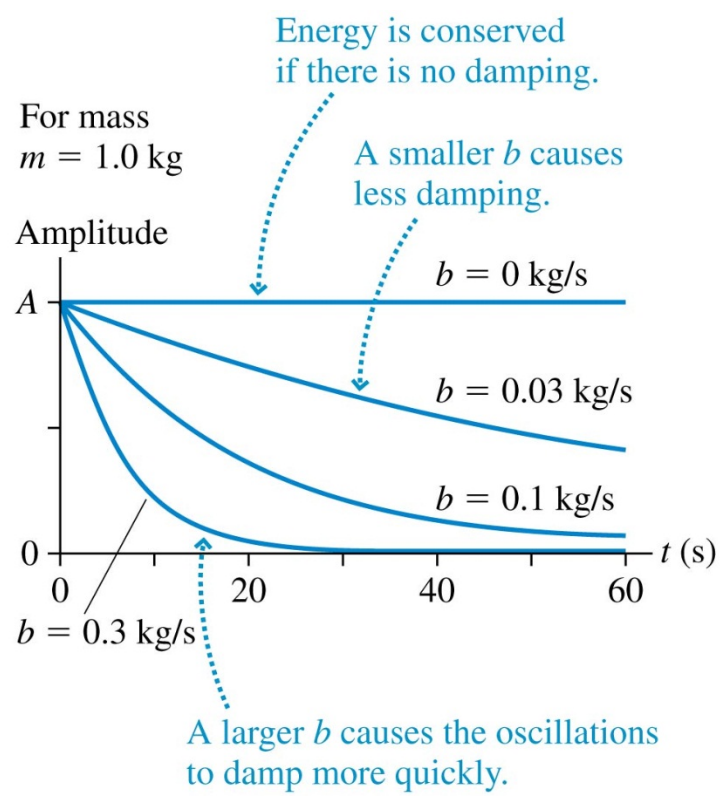


Damped Oscillations

- A damped oscillator has position $x = x_{max}\cos(\omega t + \phi_0)$, where:

$$x_{max}(t) = Ae^{-bt/2m}$$

- This slowly changing function x_{max} provides a border to the rapid oscillations, and is called the **envelope**.
- The figure shows several oscillation envelopes, corresponding to different values of the damping constant b.



Energy in Damped Systems

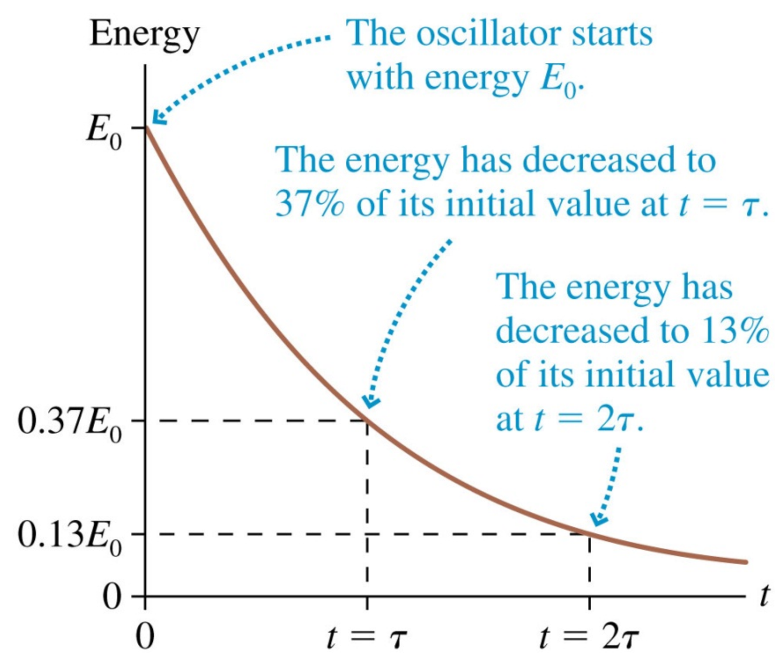
- Because of the drag force, the mechanical energy of a damped system is no longer conserved.
- At any particular time we can compute the mechanical energy from:

$$E(t) = \frac{1}{2}k(x_{max})^2 = \frac{1}{2}k(Ae^{-t/2\tau})^2 = \left(\frac{1}{2}kA^2\right)e^{-t/\tau} = E_0 e^{-t/\tau}$$

- Where the decay constant of this function is called the **time constant** τ, defined as:

$$\tau = \frac{m}{b}$$

- The oscillator's mechanical energy decays exponentially with time constant τ.



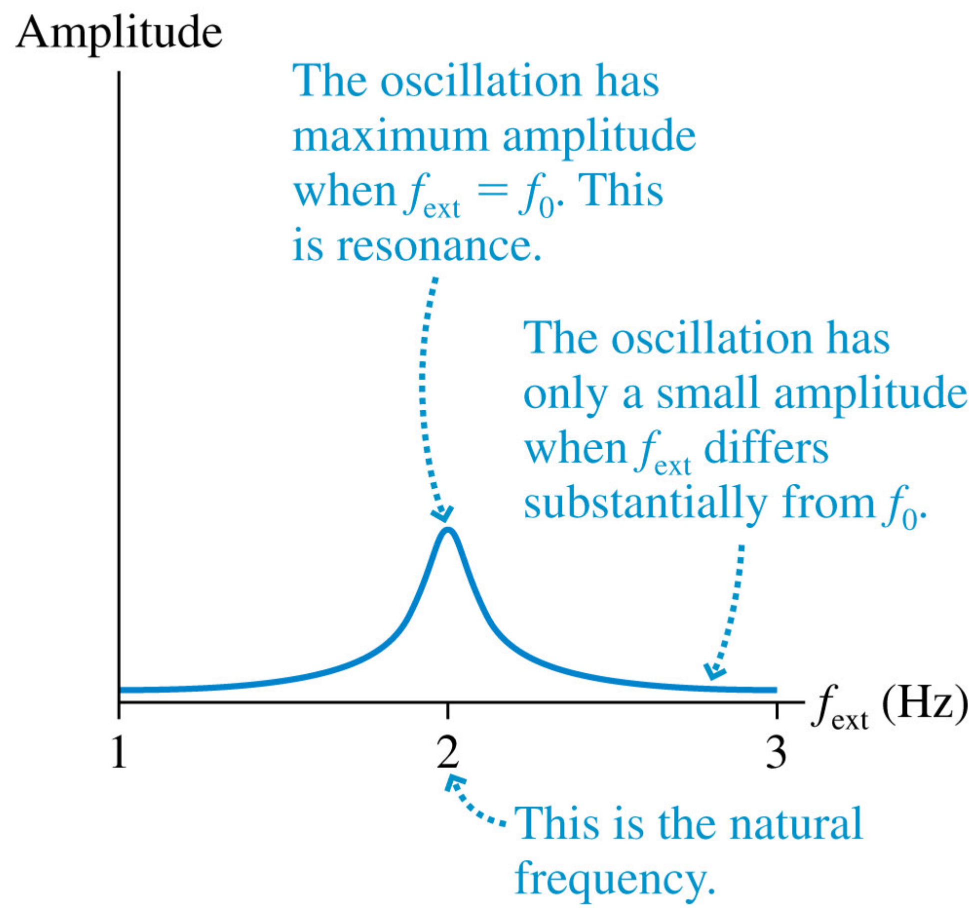
Driven Oscillations and Resonance

- Consider an oscillating system that, when left to itself, oscillates at a **natural frequency** f_0.
- Suppose that this system is subjected to a *periodic* external force of **driving frequency** f_{ext}.
- The amplitude of oscillations is generally not very high if f_{ext} differs much from f_0.
- As f_{ext} gets closer and closer to f_0, the amplitude of the oscillation rises dramatically.

A singer or musical instrument can shatter a crystal goblet by matching the goblet's natural oscillation frequency.

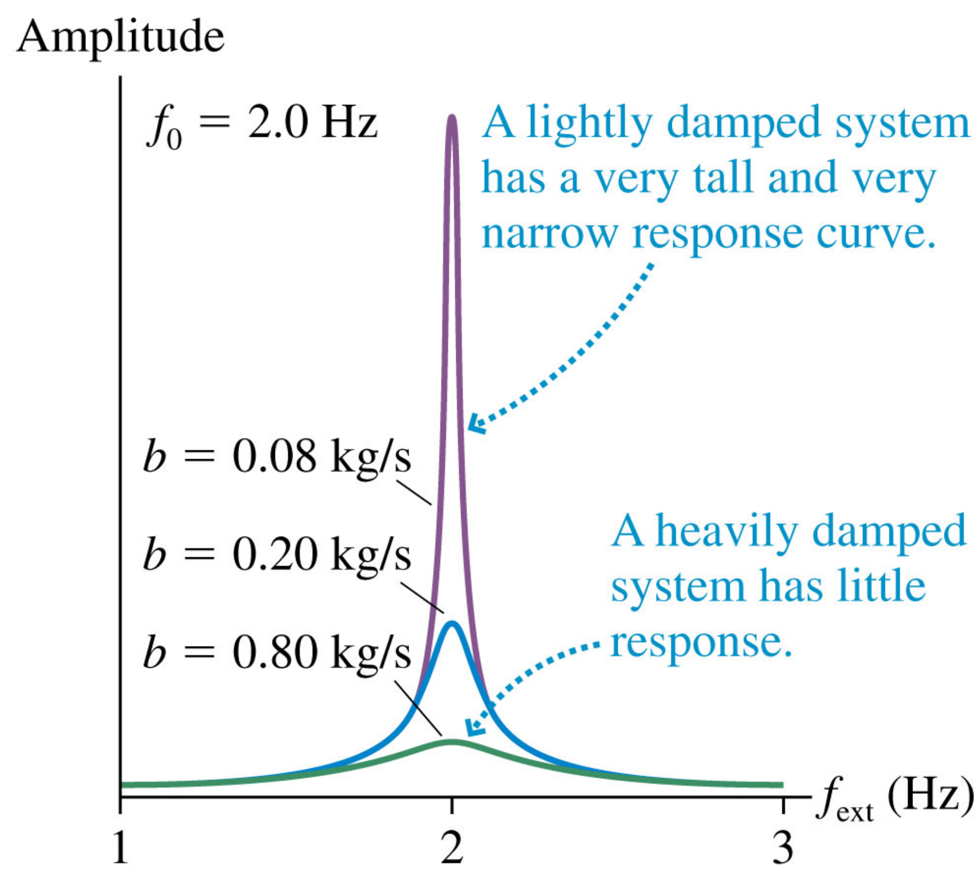
Driven Oscillations and Resonance

The response curve shows the amplitude of a driven oscillator at frequencies near its natural frequency of 2.0 Hz.



Driven Oscillations and Resonance

- The figure shows the same oscillator with three different values of the damping constant.
- The resonance amplitude becomes higher and narrower as the damping constant decreases.



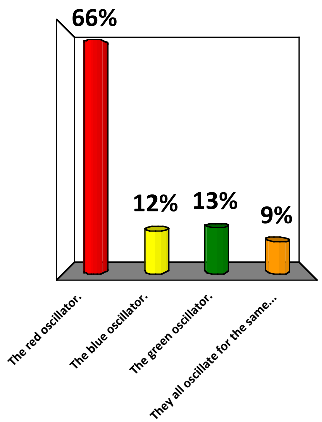
The graph shows how three oscillators respond as the frequency of a driving force is varied. If each oscillator is started and then left alone, which will oscillate for the longest time?

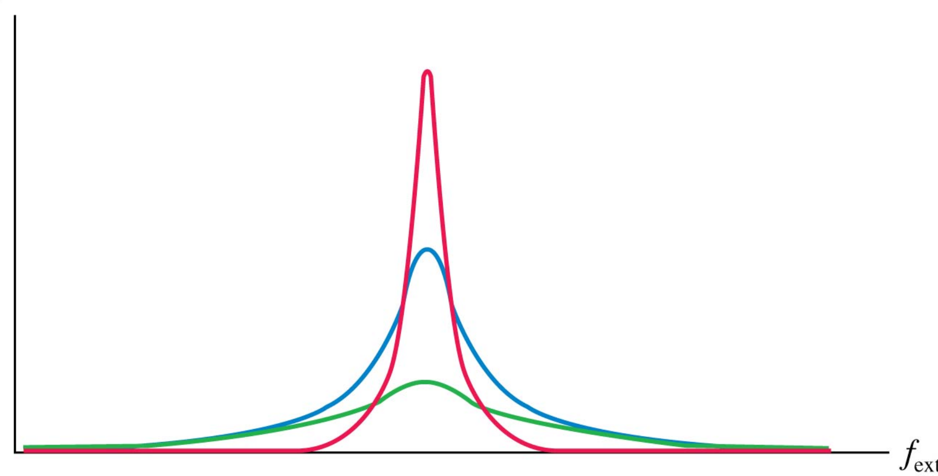
A. The red oscillator.

B. The blue oscillator.

C. The green oscillator.

D. They all oscillate for the same length of time.





- https://www.youtube.com/watch?v=xox9BVSu7Ok