MAXWELL'S EQUATIONS: RADIATION → LIGHT

To summarize, the field Equations derived from Experiments are:

GAUSS' LAW FOR COULOMB $\underset{\rightarrow}{E}$

$$\Sigma_C \underset{\rightarrow}{E} \cdot \Delta \underset{\rightarrow}{A} = \frac{1}{\varepsilon_0} \Sigma Q_i \quad (1)$$

GAUSS' LAW FOR $\underset{\rightarrow}{B}$

$$\Sigma_C \underset{\rightarrow}{B} \cdot \Delta \underset{\rightarrow}{A} \equiv 0 \quad (2)$$

LENZ'S LAW

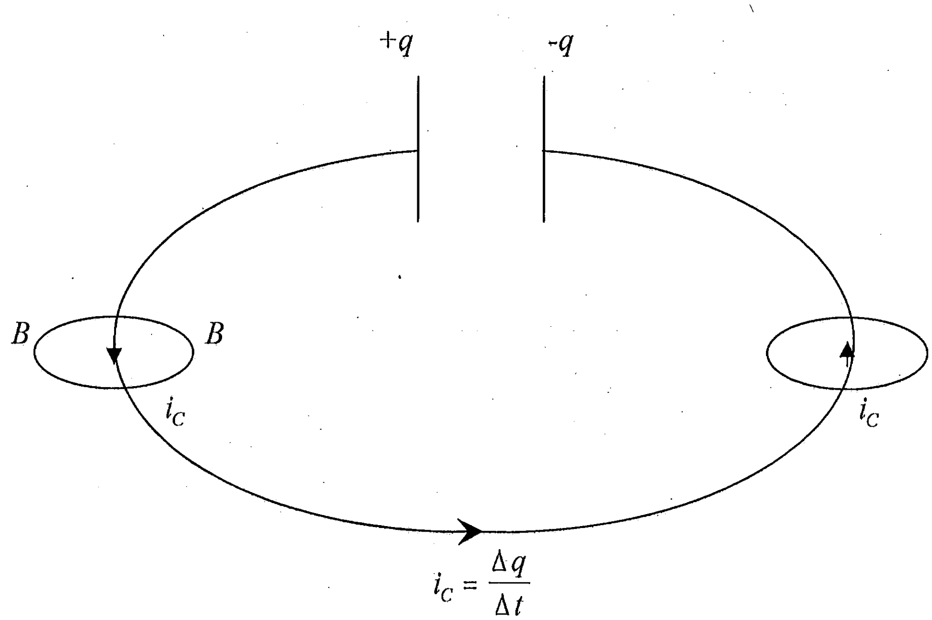
$$\Sigma_C \underset{\rightarrow}{E_{NC}} \cdot \Delta \underset{\rightarrow}{l} = -\frac{\Delta \Phi_B}{\Delta t} \quad (3)$$

AMPERE'S LAW

$$\Sigma_C \underset{\rightarrow}{B} \cdot \Delta \underset{\rightarrow}{l} = \mu_0 \Sigma I_i \quad (4')$$

When Maxwell began to study these equations, he realized that there was a serious problem. Scientists believe that at its most fundamental level nature must be symmetric.

Maxwell noticed that whereas a time varying flux of $\underset{\rightarrow}{B}$ gave rise to an $\underset{\rightarrow}{E}$ -field $\left[\underset{\rightarrow}{E_{NC}} \text{ in } Eq.(3)\right]$ there was no corresponding term in *Eq. (4')*. He immediately asserted that the above field equations could not be regarded as being complete. This was a FUNDAMENTAL PROBLEM Maxwell also noted a "PRACTICAL PROBLEM" in using *Eq. (4)*. Imagine that we charge a capacitor to $\pm q$ and then connect a wire between the two plates as shown.



It is clear that a conduction current $\frac{\Delta q}{\Delta t}$ begins to flow through the wire and so [using *Eq. (4')*] it must create a $\vec{B}$-field encircling the wire as shown. However, as soon as you cross one of the capacitor plates, both the current and $\vec{B}$ must be zero. Again, Maxwell asserted that such a discontinuity cannot be physically meaningful.

To resolve the fundamental problem Maxwell postulated that if the flux of $\vec{E}$ varies with time it must be equivalent to a current. He called this new type of current a displacement current and introduced the definition $i_D = \varepsilon_0 \frac{\Delta \phi_E}{\Delta t}$ (5)

Of course, *Eq. (4')* implies that every current generates a $\vec{B}$ so Maxwell "completed" *Eq. (4')* by writing $\Sigma_C \vec{B} \cdot \Delta \vec{l} = \mu_0 \Sigma I_C + \mu_0 \varepsilon_0 \frac{\Delta \phi_E}{\Delta t}$ (4)

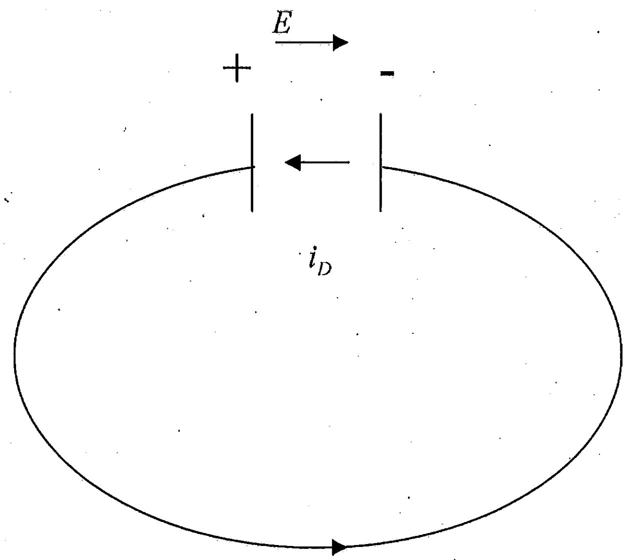
Where I_C explicitly signifies a conduction current = flow of charge in a conductor while the second term on the right comes from i_D [*Eq.* (5)].

Let us see if introduction of i_D also solves the practical problem. If the capacitor plates have an area A the $\vec{E}$-field between them is

$$\vec{E} = \frac{q}{\varepsilon_0 A}\hat{x}, \; \vec{A} = A\hat{x}$$

so $\Phi_E = \frac{q}{\varepsilon_0}$

and $i_D = \varepsilon_0 \frac{\Delta \Phi_E}{\Delta t} = \frac{\Delta q}{\Delta t} = i_C$!



[i_D is from –ive to +ive because of $\frac{\Delta q}{\Delta t}$ is –ive]

Since $i_D = i_C$ we will have no discontinuity in either the current or the $\vec{B}$-field on crossing the capacitor plate.

Maxwell has solved both the fundamental and the practical problem by proposing *Eq. (5)*.

MAXWELL'S EQUATIONS

GAUSS' Law for Coulomb $\underrightarrow{E}$:

Since a stationary charge generates a Coulomb $\underrightarrow{E}$ field, the TOTAL flux of $\underrightarrow{E}_{Cloul}$ THROUGH a closed surface is determined solely by the charges located in the volume enclosed by that surface.

$$\Sigma_C \underrightarrow{E}_{Coul} \bullet \underrightarrow{\Delta A} = \frac{1}{\varepsilon_0} \Sigma Qi \qquad (1)$$

GAUSS' Law for $\underrightarrow{B}$:

Since the elementary generators of $\underrightarrow{B}$ are point magnetic dipoles the TOTAL flux of $\underrightarrow{B}$ THROUGH a closed surface is always ZERO:

$$\Sigma_C \underrightarrow{B} \bullet \underrightarrow{\Delta A} \equiv 0 \qquad (2)$$

FARADAY - LENZ Law:

If the flux of $\underrightarrow{B}$ varies with time a Non-Coulomb $\underrightarrow{E}$ field will appear in every closed "loop" surrounding the region where the flux of $\underrightarrow{B}$ is varying. The sense of $\underrightarrow{E_{NC}}$ is invariably such as to oppose the variation in the flux of $\underrightarrow{B}$ that causes it. Hence, circulation of $\underrightarrow{E_{NC}}$ around a close loop is determined by the time rate of change of flux of $\underrightarrow{B}$ through the area within the loop; [Note: Crucial negative sign]:

$$\Sigma \underrightarrow{E_{NC}} \bullet \underrightarrow{\Delta l} = \frac{-\Delta\Phi_B}{\Delta t} \qquad (3)$$

MAXWELL - AMPERE Law:

Every current generates a $\underrightarrow{B}$ field that circulates around it. There are two types of current: (i) Conduction current which involves flow of charge in a conductor and (ii) displacement current which arises when flux of $\underrightarrow{E}$ field varies with time. Hence, Circulation of $\underrightarrow{B}$ around a closed loop is determined by currents threading the surface on which the loop is drawn.

$$\Sigma_C \underrightarrow{B} \bullet \underrightarrow{\Delta l} = \mu_0 \Sigma I_c + \mu_0 \varepsilon_0 \frac{\Delta\Phi_E}{\Delta t} \qquad (4)$$

CAUTION: i_D exists in vacuum. It never involves flow of charge. No conduction current can exist inside the capacitor!!!

Maxwell's *Equations (1)* through *(4)* have profound consequences. Let us recall his work using these in outer space, where there is vacuum, $q=0$, $i_C = 0$ so the Equations become:

$$\Sigma_C \vec{E} \cdot \Delta \vec{A} = 0 \qquad I$$

$$\Sigma_C \vec{B} \cdot \Delta \vec{A} = 0 \qquad II$$

$$\Sigma_C \vec{E}_{NC} \cdot \Delta \vec{l} = -\frac{\Delta \phi_B}{\Delta t} \qquad III$$

$$\Sigma_C \vec{B} \cdot \Delta \vec{l} = \mu_0 \varepsilon_0 \frac{\Delta \phi_E}{\Delta t} \qquad IV$$

and now indeed there is total symmetry with respect to $\vec{E}$ and $\vec{B}$. This is what led Maxwell to propose that rather than think of $\vec{E}$ and $\vec{B}$ fields, one should think of a single entity:

Electromagnetic or EM field

And call *Equations I* through *IV*, EM field Equations. He next used these Equations to predict that in vacuum there must exist EM-waves! He was able to show that the structure of these Equations is such that both the $\vec{E}$ and $\vec{B}$ have the functional form (propagation along x for example) $f(x \pm ct)$. That is, they propagate as an Electromagnetic wave with the enormous speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \, m/s$. This was a giant step forward: Maxwell had solved the problem of the nature of Radiation or Radiant energy. $\Rightarrow$ Radiation is an Electromagnetic wave. Our observable universe = Matter + Radiation

Incidentally, Einstein demonstrated that matter and radiation convert into one another there by further simplifying our picture of the universe.

→ *Heat*

→ *Light*

→ *x – rays*

→ *radiowaves*

are all cases of EM waves. They are distinguished only by their frequencies (or wavelengths). (see below)

Periodic EM Waves

As before a periodic EM wave will be represented by

$$\underset{\rightarrow}{E} = \underset{\rightarrow}{E_m} \sin\left(\frac{2\pi x}{\lambda} \pm \frac{2\pi t}{T}\right) = \underset{\rightarrow}{E_m} \sin(kx - \omega t)$$

-(1)

and

$$B = \underset{\rightarrow}{B_m} \sin\left(\frac{2\pi x}{\lambda} \pm \frac{2\pi t}{T}\right) = \underset{\rightarrow}{B_m} \sin(kx - \omega t)$$

with

$$k = \frac{2\pi}{\lambda}, \quad \omega = \frac{2\pi}{T} = 2\pi f$$

-(2)

and the speed $v = \lambda f$ or $\omega = vk$

-(3)

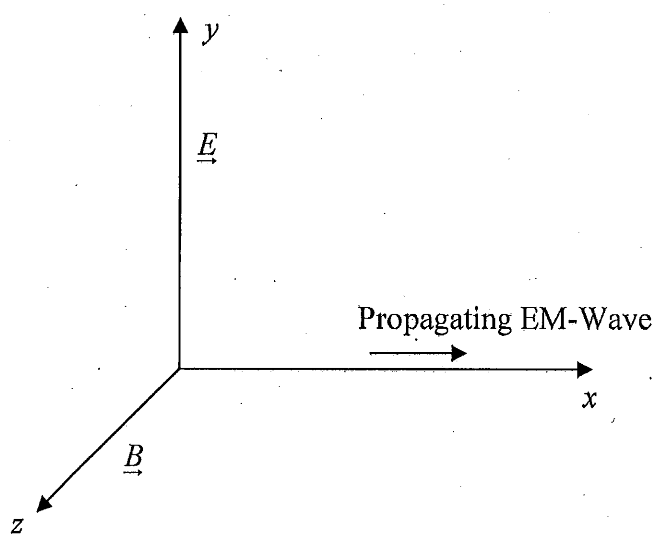
In vacuum EM-waves are totally transverse:

$$\underset{\rightarrow}{E}_m \perp \hat{x}$$

$$\underset{\rightarrow}{B}_m \perp \hat{x}$$

$$\underset{\rightarrow}{E}_m \perp \underset{\rightarrow}{B}_m$$

Indeed for a wave travelling in positive x direction $\underset{\rightarrow}{E}_m \parallel \hat{y}$ and $\underset{\rightarrow}{B}_m \parallel \hat{z}$



In vacuum EM-waves have an enormous speed (symbol c)

$$c = 3\times10^{8}\, m/\text{sec}. \tag{4}$$

In vacuum the $\underset{\rightarrow}{E}$ and $\underset{\rightarrow}{B}$ fields are related by the equation

$$E = cB \tag{5}$$

The EM wave also transports energy because energy is stored in the $\underset{\rightarrow}{E}$ and $\underset{\rightarrow}{B}$ fields. Earlier, we have proved that per m^3 the fields carry the energies

$$\eta_E = \frac{1}{2}\varepsilon_0 E^2 \tag{6}$$

$$\eta_B = \frac{B^2}{2\mu_0} \tag{7}$$

Here, ε_0 and μ_0 are constants, roughly,

$$\varepsilon_0 = 9\times10^{-12}\, F/m \tag{8}$$

(ε_0 $\quad Q^2M^{-1}L^{-3}T^{+2}$ $\quad F/m$ $\quad$ Scalar)

$$\mu_0 = 4\pi\times10^{-7}\, H/m \tag{9}$$

(μ_0 $\quad MLQ^{-2}$ $\quad H/m$ $\quad$ Scalar)

It is notable that speed of EM wave is

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

so that, because of Eq(5), in an EM wave

$$\eta_E = \eta_B \tag{10}$$

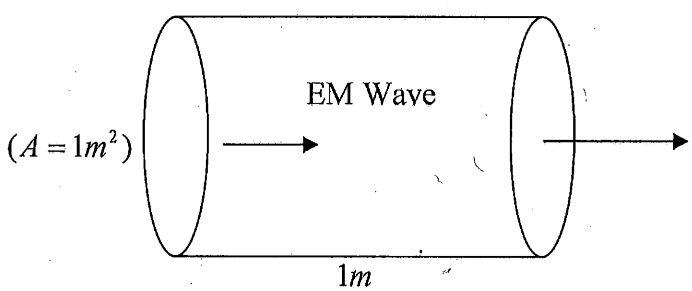
hence 1 m^3 of an EM-wave carries the energy

$$\eta_{EM} = \eta_E + \eta_B = \varepsilon_0 E^2 = \frac{B^2}{\mu_0} \tag{11}$$

As in the case of sound we can calculate the intensity of an EM-wave by using (11).

Intensity = Energy Transported per Unit Area per Unit Time

So imagine a tube of cross-sectional area 1 m^2 "filled" with an EM wave.



If its length is $1m$ then at any instant the energy stored in it is

$$\eta_{EM} = \varepsilon_0 E^2 = \frac{B^2}{\mu_0} \tag{12}$$

Where $\quad E^2 = E_m^{\ 2}\sin^2(kx - wt)$

$\qquad\quad B^2 = B_m^{\ 2}\sin^2(kx - wt)$

The average value of the energy will be

$$< \eta_{EM} > = \varepsilon_0 E_m^{\ 2} < \sin^2(kx - wt) > \qquad (13)$$

$$= \frac{B^2}{\mu_0} < \sin^2(kx - wt) >$$

But $< \sin^2(\ ) > = \frac{1}{2}$

So

$$< \eta_{EM} > = \frac{\varepsilon_0 E_m^{\ 2}}{2} = \frac{B_m^{\ 2}}{2\mu_0} \qquad (14)$$

In one second this energy will travel by c meters so energy transport per m^2 per second becomes

$$< I > = \frac{c\varepsilon_0 E_m^{\ 2}}{2} = \frac{cB_m^{\ 2}}{2\mu_0} \qquad (15)$$

From a practical point of view, if a point source of EM waves emits P joules/sec the intensity at a distance r will be

$$I = \frac{P}{4\pi r^2} (watt / m^2)$$

exactly as noted earlier for sound.

Spectrum of EM-Waves – Light

EM waves are essentially ubiquitous. The following table illustrates this point succinctly.

Name	Frequency	Wavelength (in vacuum)
AM Radio	100 kHz	kms
FM Radio	100 MHz	3m
TV - *uHF*	300 MHz	1m
Microwaves	1-100 GHz	0.1m – 0.003m
Infrared (Heat Radiation)	$10^{12} - 10^{13}$ Hz	10^{-5} m
→ Light	$10^{14} - 10^{15}$ Hz	400nm – 700nm
UV	$10^{16} - 10^{17}$ Hz	100nm
X-rays	10^{18} Hz	1nm
γ-rays	10^{20} Hz	1pm

To Summarize:
What is light?: Light is a transverse EM wave whose wavelength in vacuum (air) lies between 400nm and 700nm and speed in vacuum is $3 \times 10^8 m/\sec$.

Wave on a string: Power

$$P = \frac{1}{2}\mu A^2\omega^2 v$$

$$v = \sqrt{\frac{F}{\mu}}$$

[Average Energy stored per unit length multiplied by velocity]

Sound: Intensity

$$I = \frac{1}{2}\rho_0 S_m{}^2\omega^2 v$$

$$v = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

[Average Energy stored per unit volume multiplied by velocity]

EM-wave Light: Intensity

$$I = \frac{B_m{}^2}{2\mu_0}c = \frac{1}{2}\epsilon_0 E_m{}^2 c$$

$$c = \sqrt{\frac{1}{\mu_0\varepsilon_0}}$$

[Average Energy stored per unit volume multiplied by velocity]

In general, the propagation of a light wave is best visualized by using a construct due to Huygens'. As the light waves spread out of a point source at some time later they essentially form a spherical "wave front", a surface of constant phase. Huygens' proposed that one should treat each point of the wave front as a point source of light from which spherical wavelets emanate and a spherical surface tangent to all the wavelets locates the new wave front at a later instant. This is shown schematically in the figure. The direction of propagation is along the normal to the wave front – radical for a point source.