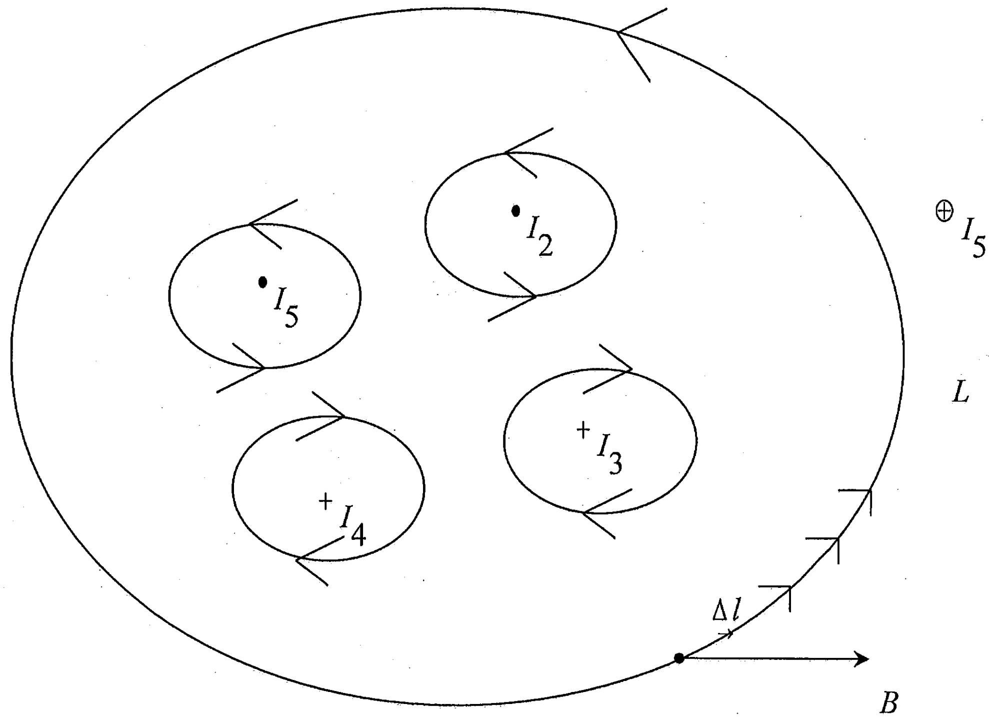
Ampere's Law and its Applications

Consider the situation shown schematically in the diagram. Currents I_1, I_2, I_3, I_4, I_5 are flowing out of $(\cdot)$ or $(+)$ into the paper. The corresponding $\vec{B}$-fields swirl around their sources as shown. The main point is that the $\vec{B}$-field lines circulate around the currents. Choose a closed loop (L). Start at A, measure B, choose a small step $\Delta \vec{l}$ along the loop. Calculate Dot product (component of $\vec{B}$ along $\Delta \vec{l}$ multiplied by Δl)

$$\vec{B} \bullet \Delta\vec{l} = B\Delta l \cos(\vec{B}, \Delta\vec{l})$$

If $\vec{B} \perp \Delta\vec{l}$, $\vec{B} \bullet \Delta\vec{l} = 0$.



Repeat this calculation at every step as shown. $\vec{B_1} \bullet \Delta\vec{l_1} + \vec{B_2} \bullet \Delta\vec{l_2} + \vec{B_3} \bullet \Delta\vec{l_3} + \ldots\ldots$
Write out the sum

$$\Sigma_c \vec{B} \cdot \Delta\vec{l};$$

c: closed loop.

This sum is called the circulation of $\vec{B}$ around a closed loop and Ampere's Law says that it is determined solely by currents threading through the surface on which the loop is drawn & only currents within the loop contribute, i.e. exclude I_5. The mathematical Equation is:

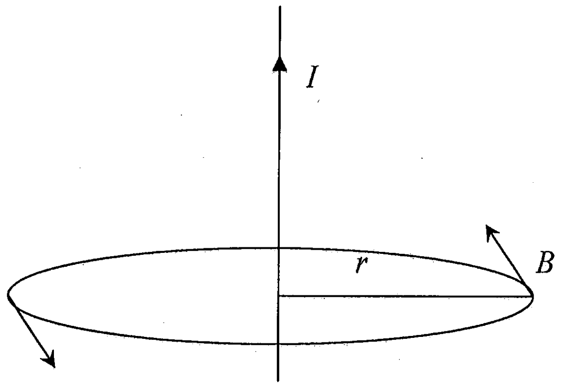
$$\Sigma_c \vec{B} \cdot \Delta\vec{l} = \mu_0 \Sigma I_i, \mu_0 = 4\pi \times 10^{-7} \frac{T-m}{A}$$

In words, circulation of $\underset{\rightarrow}{B}$ around a closed loop is proportional to the algebraic sum of the currents threading the loop surface on which the loop is drawn.

Note: As in the case of Gauss' Law, Ampere's Law gives circulation but not $\underset{\rightarrow}{B}$. To get $\underset{\rightarrow}{B}$ you need high symmetry!

Applications

1.) Single Current



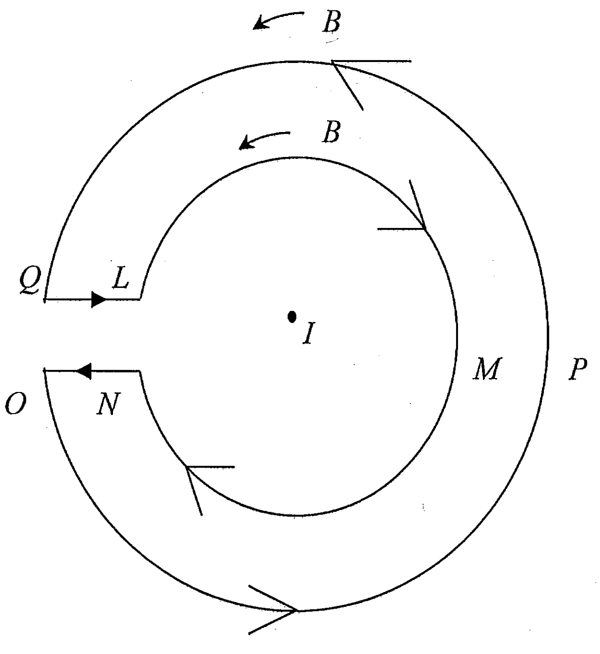
Single wire with current I, there is cylindrical symmetry so $\underset{\rightarrow}{B}$ can be a function of r only & must encircle I. [$\underset{\rightarrow}{B}$ and $\underset{\rightarrow}{\Delta l}$ are parallel to one another.]
Appropriate loop is circle of radius r centered on the wire
Hence,

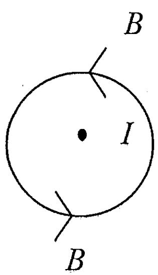
$$\sum_{c} \underset{\rightarrow}{B} \bullet \underset{\rightarrow}{\Delta l} = B \cdot 2\pi r = \mu_0 I$$

$$\text{so, } \underset{\rightarrow}{B} = \frac{\mu_0 I}{2\pi r}\hat{\varnothing}$$

as claimed previously

2.) Next, we begin by showing that if the current is outside the loop it contributes nothing to the circulation. Choose $L\ M\ N\ O\ P\ Q$ with I at the center of the circles of radii r_1, and r_2





Start at L and go around the closed "loop".

$L \to M \to N \to O \to P \to Q \to L$

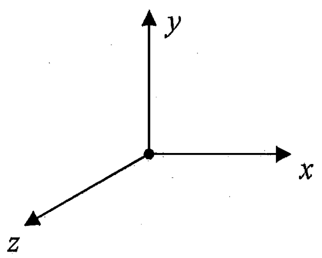
First note that $\quad \underset{\to}{B}(r_1) = \dfrac{\mu_0 I}{2\pi r_1}\hat{\Phi}$

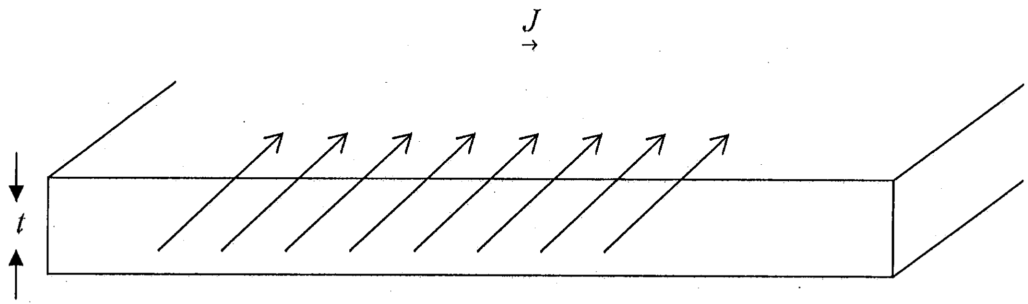
And $\quad \underset{\to}{B}(r_2) = \dfrac{\mu_0 I}{2\pi r_2}\hat{\Phi}$

$$\Sigma_C \underset{\to}{B} \bullet \underset{\to}{\Delta l} = \underbrace{\frac{\mu_0 I}{2\pi r_1} 2\pi r_1 \cos 180^\circ}_{L \to N} + \underbrace{0}_{N \to O} + \underbrace{\frac{\mu_0 I}{2\pi r_2} 2\pi r_2 \cos 0}_{O \to Q} + \underbrace{0}_{Q \to L} = 0$$

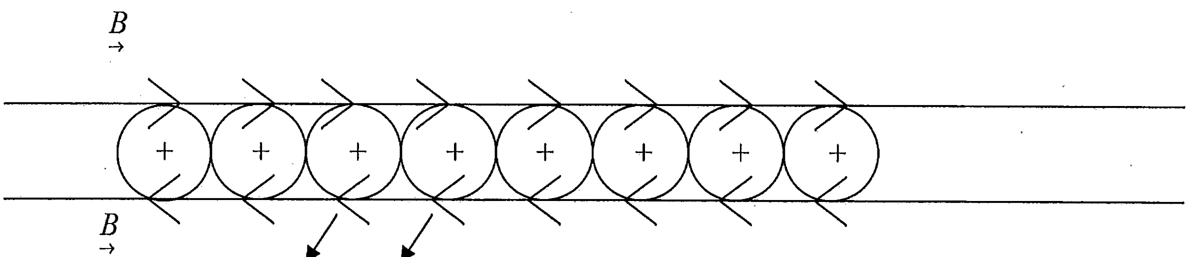
For $N \to O$ and $Q \to L$, $\underset{\to}{B} \perp \underset{\to}{\Delta l}$ hence $\underset{\to}{B} \bullet \underset{\to}{\Delta l} = 0$.

3.)

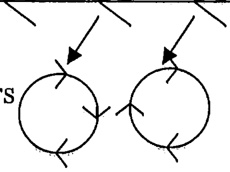




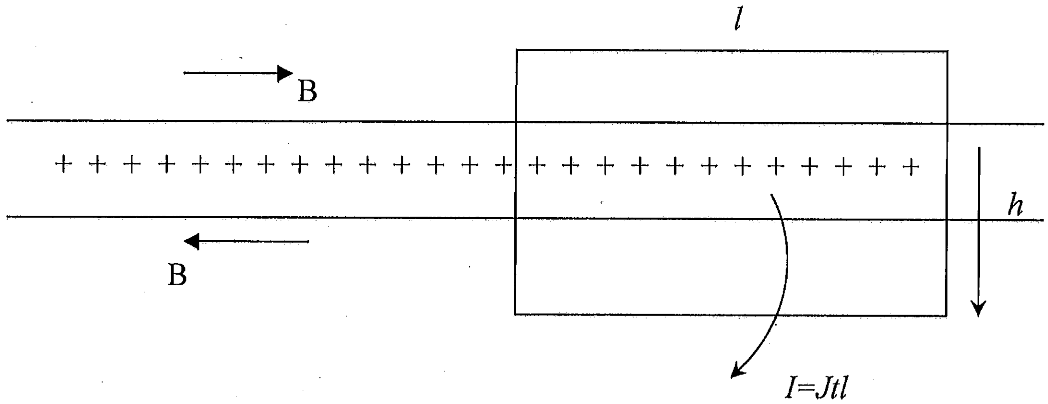
At y=0 there is a Current sheet of thickness t carrying current density $\underset{\to}{J} = -J\hat{z}$. Looking it end-on we see sources of $\underset{\to}{B}$ as



Any two neighbors



and we see that y-components of $\vec{B}$ cancel out. $\vec{B} \| \hat{x}$ survives. Let us take loop of width l and height h.

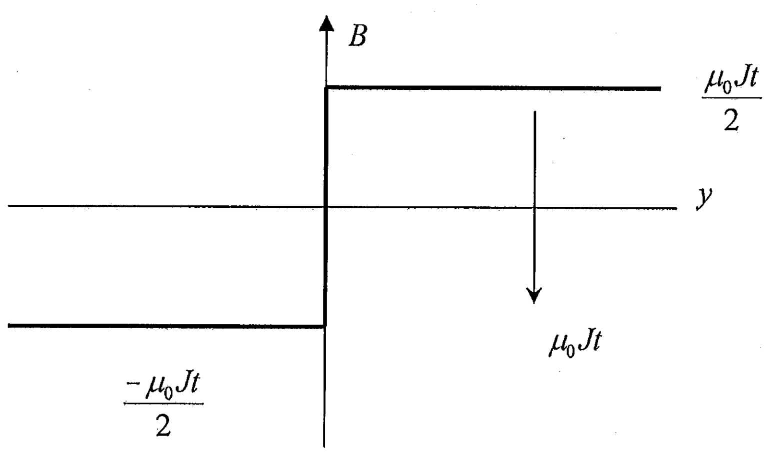


$$\Sigma_c \vec{B} \cdot \Delta \vec{l} = Bl + 0 + Bl + 0$$
$$= 2Bl$$
$$= \mu_0 Jtl$$

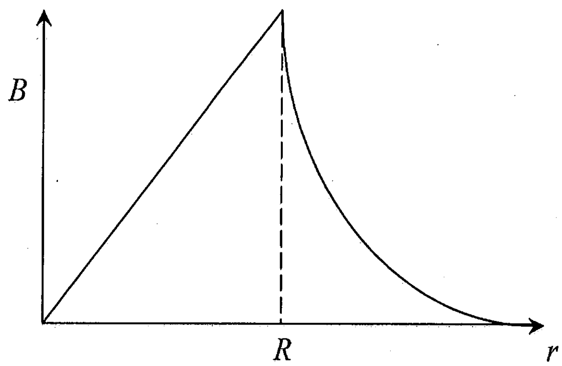
so,

$$B = \frac{\mu_0 Jt}{2} \;\&\; \vec{B} = \frac{\mu_0 Jt}{2}\hat{x} \qquad y > 0$$
$$= \frac{-\mu_0 Jt}{2}\hat{x} \qquad y > 0$$

That is, $\vec{B}$-field will jump by $\mu_0 Jt$ on crossing the current sheet from y<0 to y>0.

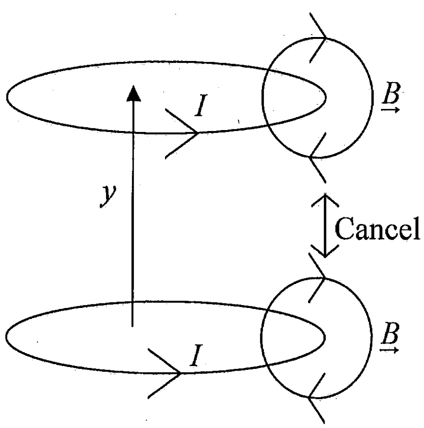


For $>R$, entire I contributes $\underset{\rightarrow}{B} = \frac{\mu_0 I}{2\pi r}\hat{\varnothing}$



6.) Solenoid: Tightly wound, small radius, length much larger than radius:

N turns, L long $n = \frac{N}{L} =$ # of turns per meter.
Look at two neighboring turns



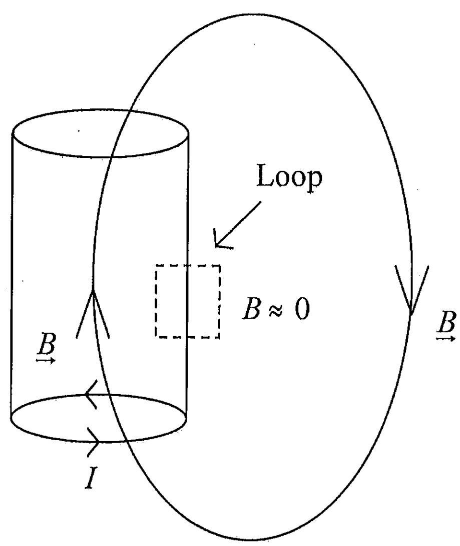
r-component cancels
B_y inside survives.

Long-narrow solenoid. [$\underset{\rightarrow}{B}$ field lines must come out of the top, loop around and enter at bottom with no breaks or bends allowed.]

$\underset{\rightarrow}{B} \approx 0$ just outside.

Take loop as shown $\begin{array}{l} Bl = \mu_0 nIl \\ B = \mu_0 nI \end{array}$

For case shown $\underset{\rightarrow}{B} = \mu_0 nI\hat{y}$

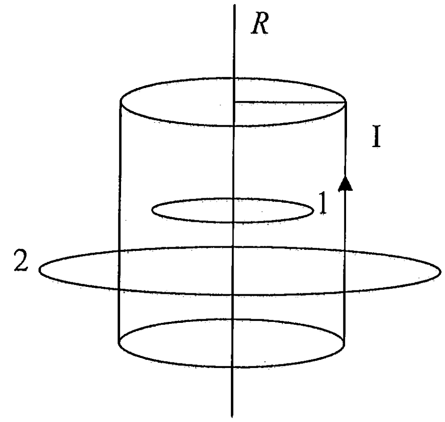


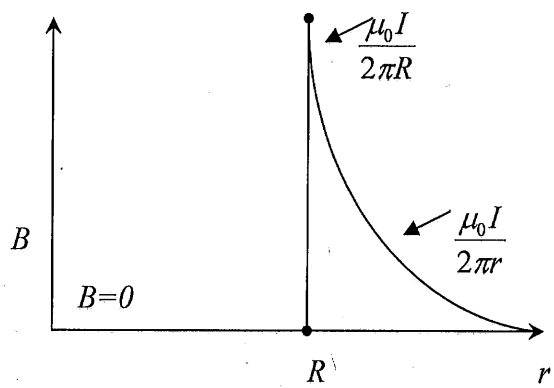
4.) Hollow Cylindrical Conductor- Radius R, carries uniform current. We want $\vec{B}$ at a distance r from its axis. Since there is a cylindrical symmetry we should use circles centered on the axis for our closed loop.
For $r<R$, use loop 1.

$B \cdot 2\pi r = 0$. No Current threads through loop 1.

for $r > R$, $B \cdot 2\pi r = \mu_0 I$, the entire current threads loop 2.

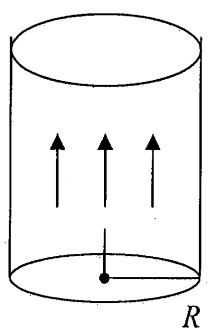
so, $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varnothing}$





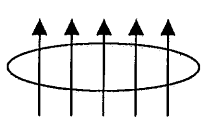
Note: if cylinder has wall thickness t: $I = J \cdot 2\pi Rt$ and field at surface would be $\mu_0 Jt$.
Again field would jump by $\mu_0 Jt$ on crossing a current sheet.

5.) SOLID CYLINDRICAL CONDUCTOR – with uniform current



Define $J = \frac{I}{\pi R^2}$

Now for $r<R$ $I = J\pi r^2$



$B \cdot 2\pi r = \mu_0 J\pi r^2$

$\vec{B} = \frac{\mu_0 Jr}{2}\hat{\varnothing}$ $\quad r < R$