LINEAR HARMONIC MECHANICAL OSCILLATORS
(Translation)

Definition: A mass M will perform linear harmonic (or simple harmonic) oscillations about a point of equilibrium ($\underset{\rightarrow}{F} = 0$) if it is acted upon by a force which is proportional to its displacement (x) from equilibrium and always acts in a direction opposite to the displacement vector.

So essential ingredients are:

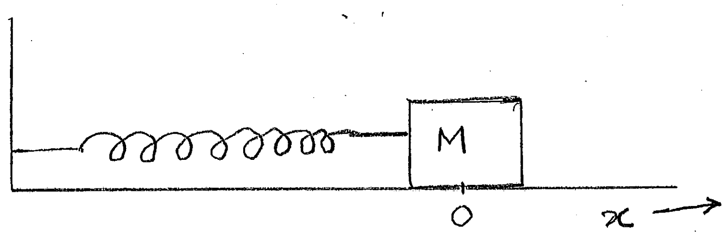
Magnitude of F proportional to (x)
Direction of F opposite to ($\underset{\rightarrow}{x}$)

Hence $\underset{\rightarrow}{F}$ is called: RESTORING FORCE

Alternate Definition: The object has a potential energy which varies as the square of the displacement.

L.H. oscillations are ubiquitous in the physical world but to start with we consider only 4 realizations of the above definition.

1. Spring-Mass Oscillator (Horizontal)

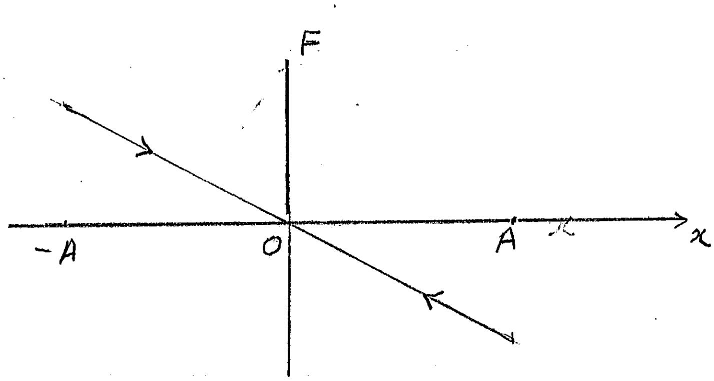


A mass M is attached to a spring of constant k. The other end of the spring is attached to a rigid post. M is placed on a smooth frictionless horizontal table such that when it is at x=0, the spring is relaxed so M is in $\equiv m$ as its weight is supported by N_R.

If we displace M by an amount x, immediately the spring force

$$\underset{\rightarrow}{F} = -kx\hat{x} \qquad (1)$$

comes into play. So if after displacing M by an amount A we let go, M will be under the influence of the force of Eq (1) and will move back and forth as a L.H. oscillator.

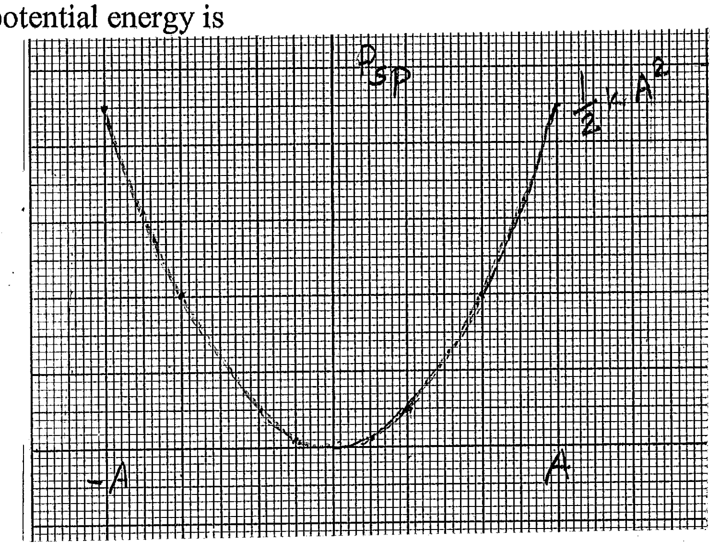


Why? When you let go, the mass experiences a force that brings it back to 0. But when it reaches 0 it has a kinetic energy and it cannot stop. So it keeps going until it gets to –*A*. But now again it has a force that wants to bring it back to 0. And there you have it. Every time it returns to $\equiv m$ it fails to stop and when it stops it is not in $\equiv m$.

The corresponding potential energy is

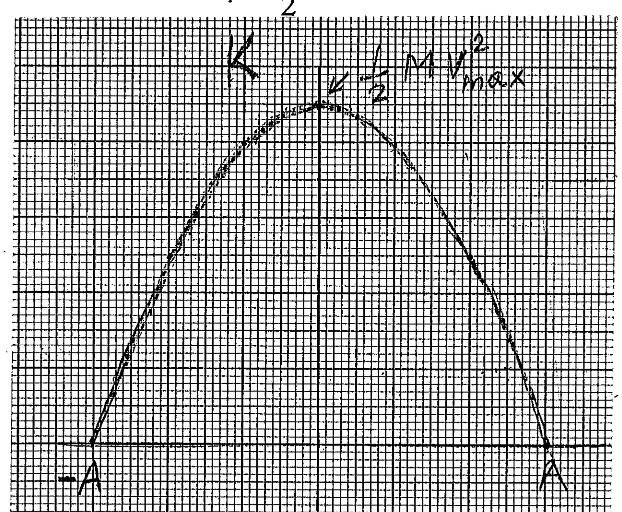
$$P_{sp} = \frac{1}{2}kx^2$$

That is:



So at x=A, $P_{sp} = \frac{1}{2}kA^2$. Let go. P wants to reduce but when M reaches 0 ($x = 0$) P is converted to kinetic energy ($\frac{1}{2}MV_{max}{}^2$) and that takes it to –A where $P_{sp} = \frac{1}{2}kA^2$ again.

Kinetic energy:



Energy conservation requires

$$\frac{1}{2}kA^2 = \frac{1}{2}MV^2_{max} = \frac{1}{2}kx^2 + \frac{1}{2}MV^2 \qquad (2)$$

where V is the speed when M is at any point x between 0 and A.

So the FORCE equation is

$$\vec{F} = -kx\hat{x}$$

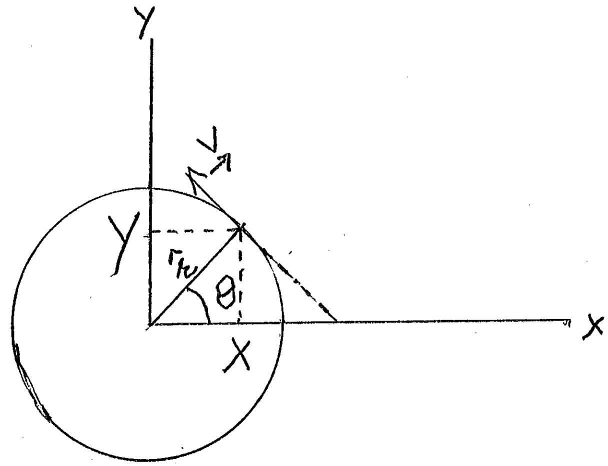
and hence acceleration is

$$\vec{a} = -\frac{k}{M}x\hat{x} \qquad (3)$$

[Never leave out the minus sign.]

Uniform Circular Motion ↔ Simple Harmonic Oscillation

Point P travels on circle of radius A at constant angular velocity $\underset{\to}{\omega} = \omega\hat{z}$ (counter-clockwise) and $\Theta = \Theta_0 + \omega t$



Position Vector
$\underset{\to}{r} = A\hat{r}$

Velocity Vector
$\underset{\to}{v} = A\omega\hat{\tau}$

Acceleration Vector
$\underset{\to}{a} = -A\omega^2\hat{r}$

First
X-component position $X = A\cos\Theta$
velocity $v_x = -A\omega\sin\Theta$
acceleration $a_x = -A\omega^2\cos\Theta = -\omega^2 X$

Compare Eq (3) $\underset{\to}{a} = \frac{-k}{M}x\hat{x}$

So as P goes around circle X mimics motion of linear harmonic oscillator with

$$\omega = \sqrt{\frac{k}{M}}$$

And we get equation (7)

y-Second component $y = A\sin\Theta$
$v_y = +A\omega\cos\Theta$
$a_y = -A\omega^2\sin\Theta = -\omega^2 Y$

and the variation of Y mimics a linear harmonic oscillator along y-axis

$$\omega = \sqrt{\frac{k}{m}}$$

$y = A\sin(\omega t + \Theta_0)$
$v = A\omega\cos(\omega t + \Theta_0)$
and $a = -A\omega^2\sin(\omega t + \Theta_0)$

As we showed above, the position, velocity (magnitude) and acceleration are given by

$$x = A\cos(\omega t + \Theta_0) \qquad (4)$$

$$V = -A\omega\, Sin\,(\omega t + \Theta_0) \qquad (5)$$

$$a = -A\omega^2\, Cos\,(\omega t + \Theta_0) = -\omega^2 x \qquad (6)$$

where A = amplitude, the largest value of x and is determined by the potential energy stored in the spring mass system to initiate the motion.
and ω = Angular frequency

$$\omega = \sqrt{\frac{k}{M}} \qquad (7)$$

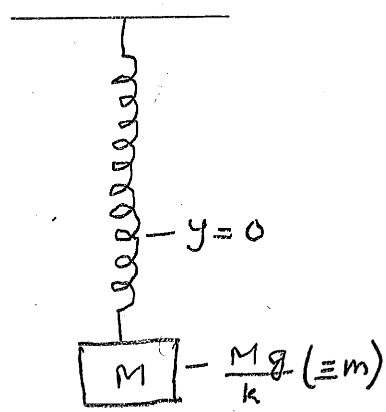
and is related to the two fundamental properties of the oscillator.

Frequency $f = \frac{\omega}{2\pi}$ = # of oscillations per second

Period $T = \frac{1}{f}$ = time taken for one complete oscillation

θo = phase and tells you precisely the position of M at $t = 0$. For example, if $\theta o = 0$, M is at A at $t=0$. If $\theta o = \pi/2$, M is at 0 at $t=0$.

2. Spring-Mass Oscillator (Vertical)



In this case, of course, as soon as you attach the mass, the spring will stretch. If you hold the mass and allow the stretching to happen while you keep holding it until the spring is fully stretched due to the force $-Mg\,\hat{y}$ and then let go, the mass will be in $\equiv m$ because spring force

$$\vec{F}_{s_p} = -ky\hat{y}$$

Will be balancing $-Mg\,\hat{y}$

$$(-ky - Mg)\,\hat{y} = 0$$

i.e.

$$\Delta y = -\frac{Mg}{k}$$

to give $\vec{F}_{s_p} \parallel +\hat{y}$

Now, if you want m to oscillate you must pull it by an amount A, store $\frac{1}{2}kA^2$ energy in it and it will oscillate around the new $\equiv m$ $pt.$ $\Delta y = \frac{-Mg}{k}$, A above and $-A$ below it.

Note that $-Mg\,\hat{y}$ is a constant force and is not a restoring force, so it does not affect the period of the oscillator. All it does is to move the point of $\equiv m$.

ω is still equal to $= \sqrt{\frac{k}{M}}$

$$T = 2\pi\sqrt{\frac{M}{k}}$$

Alternate expt.

y=0. spring unstretched
Attach M.
Let go.
The mass will drop.
Come to rest.
Rise.
Oscillate.

How far is the initial drop?
Determined by conservation of energy.

At $y = 0$ $\quad P_{sp} = 0 \quad \left[P_{sp} = \frac{1}{2}ky^2\right]$

$P_g = 0 \quad \left[P_g = Mg\,y\right]$

So now y max

$$\frac{1}{2}ky^2_{\max} + Mg\,y_{\max} = 0$$

$$y_{\max} = -\frac{2Mg}{k}$$

Of course, $= m$ pt. is still $y = -\frac{Mg}{k}$

So mass will oscillate between

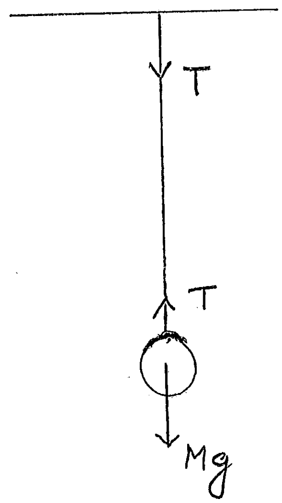
$$y = 0 \text{ and } y = \frac{-2Mg}{k}$$

And again Mg cannot be a restoring force. Only $\underset{\rightarrow sp}{F}$ can do that, so

$$\omega = \sqrt{\frac{k}{M}}$$

Very interesting observation: The entire motion of the oscillator happens while total energy (kinetic plus potential) is equal to zero.

3. SIMPLE PENDULUM



Mass M hung from a rigid support using a light string of length l [if mass has size l must be measured to its C.G.]
When string is vertical (nearly) mass in $\equiv m$

$$T - Mg = 0$$

If you pull mass sideways by amount $+\theta$ max and let go it will oscillate between the θ max *and* $-\theta$ max.

Consider the forces at some angle θ. We can break $-Mg\hat{y}$ into its components:

i. along the string (radius of circle on which M travels)
   $Mg\, Cos\, \theta$

ii. Perpendicular to string (along tangent to circle on which M travels)

$$\vec{F}_\tau = -Mg\, Sin\, \theta\, \hat{\tau}$$

Because of *–ive* sign it is certainly a restoring force, so that is good. But it is not proportional to displacement as it stands. However, if we play our cards right, we can make it so. Recall that when $\theta << 1$,

$Sin\, \theta \approx \theta$

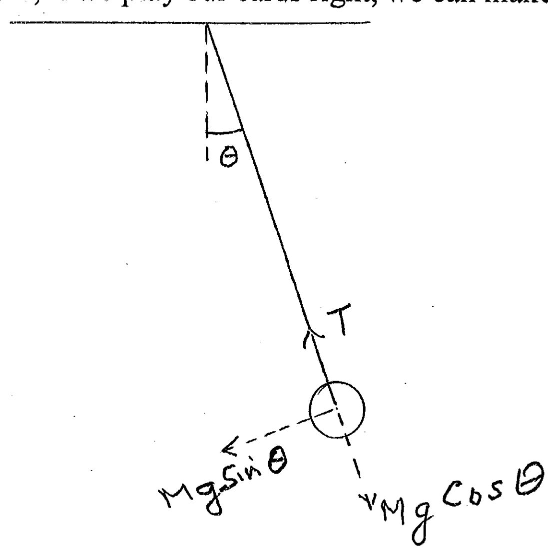
So make sure θ is small!

Then $$\vec{F}_\tau = -Mg\theta\hat{\tau}$$

And that will give us LH oscillations.

Notice: $\theta = \frac{s}{l}$

Where s is displacement along arc.



So

$$\underset{\rightarrow}{F_\tau} = \frac{-Mg\,s}{l}\hat{\tau}$$

$$\underset{\rightarrow}{a} = \frac{-g\,s}{l}\hat{\tau}$$

$$= -\omega^2 s\,\hat{\tau}$$

And therefore angular frequency must be $\omega = \sqrt{\frac{g}{l}}$

And period becomes $T = 2\pi\sqrt{\frac{l}{g}}$

As you checked in your experiment.

By analogy with case 1 above, we can write

$S = A\cos(\omega t + \alpha),\ A = l\theta \max$

Or $\theta = \theta max\ cos(\omega t + \alpha)$

$V_t = -A\ \omega \sin(\omega t + \alpha)$

$a_t = -\omega^2 A$

Also, potential energy (assuming $P = 0$, when $\theta = 0$) is

$P = Mg\,l\,(1\text{-}cos\,\theta)$

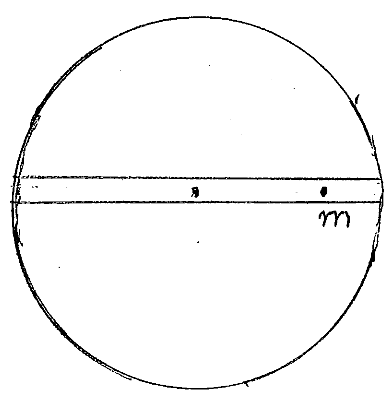
But for $\theta \ll 1$

$$Cos\theta = 1 - \frac{\theta^2}{2}$$

So, $P = \frac{Mgl\theta^2}{2}$

Which is what we need for LH oscillators

4. Sphere with a Diametric Hole



Take a uniform sphere made of a material of density d. Make a diametric hole in it (very narrow hole). In the hole, place a mass m at a distance r from the center of the sphere.

What is the gravitational force on m:

$$\vec{F}_G = -\frac{4\pi}{3} G d\, m\, r\, \hat{r}$$

It is a restoring force.
It is directly proportional to r.

If you let m go it will have LH oscillations with angular frequency

$$\omega = \sqrt{\frac{4\pi}{3} G d}$$

So there!

<u>LINEAR HARMONIC OSCILLATORS</u>
(ROTATIONAL)

Definition: To get L.H. Oscillations involving rotation we need $\underset{\rightarrow}{\tau}$ a torque which is proportional to the angular displacement (θ) from equilibrium and is opposite to the displacement vector. That is,

$$\underset{\rightarrow}{\tau} = -c\underset{\rightarrow}{\theta}$$

In our case all rotations are about z-axis so

$$\underset{\rightarrow}{\tau} = -c\theta\,\hat{z}$$

And since

$$I\underset{\rightarrow}{\alpha} = \underset{\rightarrow}{\tau}$$

$$\underset{\rightarrow}{\alpha} = -\frac{c}{I}\theta\,\hat{z}$$

which by analogy with the prior discussion gives

$$\underset{\rightarrow}{\alpha} = -\omega^2\theta\,\hat{z}$$

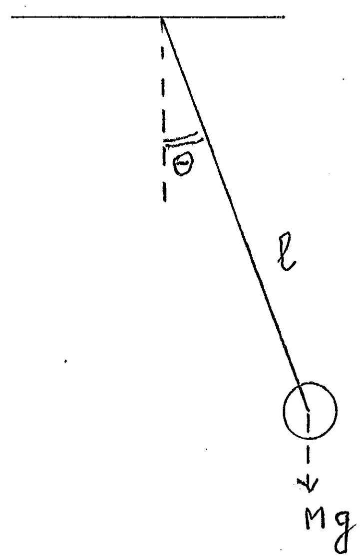
And so angular frequency

$$\omega = \sqrt{\frac{c}{I}}$$

Two examples follow, Note that Potential Energy is:

$$P_\theta = \frac{1}{2}c\theta^2 \qquad \begin{bmatrix} P_\theta = 0 \\ \theta = 0 \end{bmatrix}$$

<u>1. SIMPLE PENDULUM</u>



Single mass point M moving on a circle of radius l. At angle θ.

$$\underset{\rightarrow}{\tau} = \underset{\rightarrow}{r} \times \underset{\rightarrow}{F}$$
$$= -Mg\,l\,Sin\,\theta\,\hat{z}$$

[for case shown]

and again we need $\theta << 1$ so that

$$\underset{\rightarrow}{\tau} \cong -Mg\,l\,\theta\,\hat{z}$$

Here $I = Ml^2$

$$\text{So } \underset{\rightarrow}{\alpha} = \frac{g}{l}\theta\,\hat{z}$$

and therefore

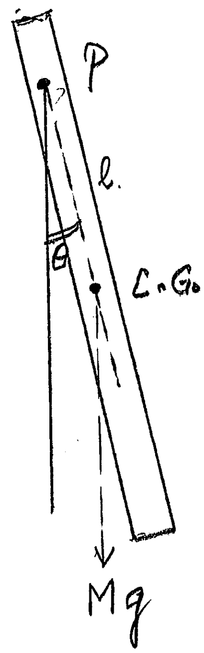
$$\omega = \sqrt{\frac{g}{l}} \text{ and } T = 2\pi\sqrt{\frac{l}{g}}$$

as before.

2. PHYSICAL PENDULUMS

Bar suspended from a rigid support at P. $C.G$ is l meters away from P.

$$\underset{\rightarrow}{\tau} = -Mg\,l\,Sin\,\theta\,\hat{z}$$



and for $\theta << 1$

$$\underset{\rightarrow}{\tau} = -Mg\,l\,\theta\,\hat{z}$$
$$= I\underset{\rightarrow}{\alpha}$$

so

$$\underset{\rightarrow}{\alpha} = \frac{-Mg\,l}{I}\theta\,\hat{z}$$

yielding

$$\omega = \sqrt{\frac{Mg\,l}{I}}$$