FORMULAE FOR WEEK 7 Problems

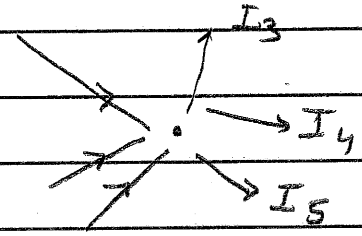
KIRCHHOFF'S RULES:

LOOP RULE Total change of potential around closed loop in a Circuit is zero.

In applying this make sure that V drops when following direction of I and rises when going against direction of I.

JUNCTION RULE

At a junction total I out is equal to total I in.



RC-Circuit Time Const. $\tau = RC$.

Charging $i = \frac{\varepsilon}{R} e^{-t/RC}$ Discharging $i = -\frac{\varepsilon}{R} e^{-t/RC}$

$V_c = \varepsilon\left[1 - e^{-t/RC}\right]$ $V_c = \varepsilon e^{-t/RC}$.

NO CURRENT BETWEEN CAPACITOR PLATES

During discharging current flows in opposite Direction

Electrical Equivalent: 4.18 J of Electrical energy mimics 1 cal of Heat.

B-field: A moving charge experiences a force in B and the force is perpendicular to its velocity at ALL Times:

$$\vec{F}_B = q\left[\vec{v} \times \vec{B}\right]$$

Homework 7 Solutions

Problem 7-1 To quote the text:

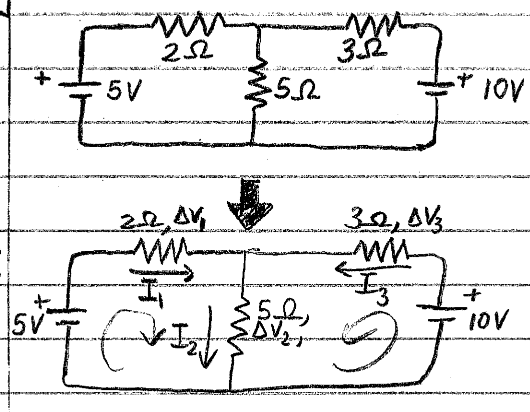
"The loop rule is based on the fact that the change of potential between two points is independent of the path [taken] because potential is derived from potential energy, and the latter is defined for a conservative force, so net change of potential on a closed loop must be zero.

(On a related note, if the loop rule didn't hold, you would be able to gain free energy by going around the loop, which would violate conservation of energy).

The junction rule is based on the fact that the flow of charge is continuous. Apart from what is involved in setting up the original field to drive a current, there can be no continuous accumulation or depletion of charge at a junction. Charge is conserved, so the flow of charge out of a junction per second must equal the flow of charge into the junction per second.

HW7 Solutions

Problem 7-2



I expect:

First, we need to have labels for all the things we'll need to calculate: currents I_1, I_2, and I_3, and potential drops ΔV_1, ΔV_2, and ΔV_3.

Common sense suggests that current loops up from the positive terminals of the batteries, around, down through the 5Ω resistor (#2), and back to the batteries.

Kirchoff's junction rule gives us an equation for the currents. I_1 and I_3 flow *into* the upper three-way junction; I_2 flows out. Currents must balance, so $\boxed{I_1 + I_3 = I_2}$. (i)

Kirchoff's loop rule gives us two equations for voltage drops, one around each loop. On the left, we have a 5V battery to be balanced out by ΔV_1 and ΔV_2. On the right, we have a 10V battery to be balanced out by ΔV_3 and ΔV_2. So:

$$\boxed{5V - \Delta V_1 - \Delta V_2 = 0} \text{ (ii)} \qquad \boxed{10V - \Delta V_3 - \Delta V_2 = 0} \text{ (iii)}$$

(Equations are labeled with Roman numerals. ΔV is taken to be positive, for bookkeeping)

Ohm's law gives us three equations relating R's and ΔV's:

$$\underset{\text{(iv)}}{\boxed{\Delta V_1 = I_1 R_1 = I_1(2\Omega)}} \quad \underset{\text{(v)}}{\boxed{\Delta V_2 = I_2 R_2 = I_2(5\Omega)}} \quad \underset{\text{(vi)}}{\boxed{\Delta V_3 = I_3 R_3 = I_3(3\Omega)}}$$

Now it becomes a matter of algebra. Rather messy algebra, sadly. I will spare you my false starts and show a path to the solution - *not* the only one.

Combine (ii) and (iii): $5V - \Delta V_1 - \Delta V_2 = 0 = 10V - \Delta V_3 - \Delta V_2$

Cancel terms to get $\quad -\Delta V_1 = 5V - \Delta V_3$

(vii) $\boxed{\Delta V_3 - \Delta V_1 = 5V}$ or $\boxed{\Delta V_1 = \Delta V_3 - 5V}$

Use (iv), (v), and (vi) to turn (i) from a current equation into a voltage equation:

$$\text{(iv)} \Rightarrow I_1 = \left(\Delta V_1 / 2\Omega\right) \qquad \text{(v)} \Rightarrow I_2 = \left(\Delta V_2 / 5\Omega\right) \qquad \text{(vi)} \Rightarrow I_3 = \left(\Delta V_3 / 3\Omega\right)$$

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HW7 Solutions

Problem 7-2 (continued)

Plug that into (i), and we get:

$$\left(\frac{\Delta V_1}{2\,\Omega}\right) + \left(\frac{\Delta V_3}{3\,\Omega}\right) = \left(\frac{\Delta V_2}{5\,\Omega}\right)$$

Multiply by $30\,\Omega$: $\boxed{15\Delta V_1 + 10\Delta V_3 = 6\Delta V_2}$ Call this (viii).

Now, with (iii), (vii), and (viii) in our hands, we have reduced the problem somewhat: we can juggle these three equations to find the ΔV's, then use Ohm's Law to easily calculate the I's.

From (vii), we know that $\Delta V_1 = \Delta V_3 - 5V$. If we plug this into (viii) we get:

$$15(\Delta V_3 - 5V) + 10\Delta V_3 = 6\Delta V_2$$
$$25\Delta V_3 - 75V = 6\Delta V_2$$
$$25(\Delta V_3 - 3V) = 6\Delta V_2$$
$$\text{(ix)} \quad \boxed{\Delta V_3 - 3V = 0.24\,\Delta V_2} \text{ or } \Delta V_3 = 0.24\,\Delta V_2 + 3V$$

But we also have (iii), which can be written: $10V = \Delta V_3 + \Delta V_2$.

Plug (ix) into this:

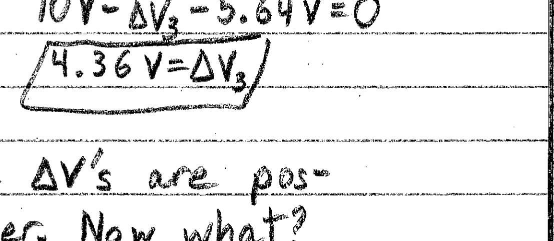
$$10V = \Delta V_2 + (0.24\,\Delta V_2 + 3V)$$
$$7V = 1.24\,\Delta V_2$$
$$\boxed{\Delta V_2 = 5.64\,V}$$

This is light at the end of the tunnel! Now we can easily use (ii) and (iii):

$$5V - \Delta V_1 - 5.64V = 0 \qquad 10V - \Delta V_3 - 5.64V = 0$$
$$\boxed{\Delta V_1 = -0.64\,V} \ldots \qquad \boxed{4.36\,V = \Delta V_3}$$

Uh-oh. On the last page I assumed all the ΔV's are positive, but here I have a negative number. Now what?

I know Ohm's Law and Kirchhoff's Laws work. I must have made a mistake. Fortunately, this particular mistake is natural and easy to fix: I was wrong about the direction of current through resistor #1!



This will probably happen to some of you, too. All I need to do is see that instead of flowing to the right, I_1 flows 'upstream' to the left, driven through the 5V battery by the bigger 10V battery.

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HW 7 Solutions

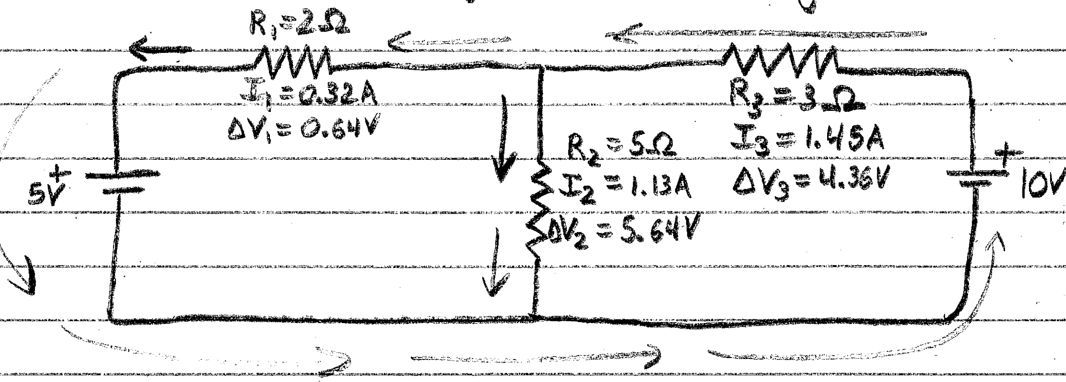
Problem 7-2 (continued) So we can simply recognize that current flows through resistor #1 in the opposite direction from what I thought, and move on. Using Ohm's Law:

$V = IR$

$I = V/R$

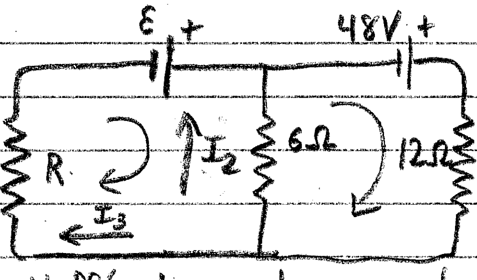
$$I_1 = \Delta V_1/(2\,\Omega) \qquad I_2 = \Delta V_2/(5\,\Omega) \qquad I_3 = \Delta V_3/(3\,\Omega)$$
$$I_1 = 0.64\,V/2\,\Omega \qquad I_2 = 5.64\,V/5\,\Omega \qquad I_3 = 4.36\,V/3\,\Omega$$
$$\underline{I_1 = 0.32\,A} \qquad \underline{I_2 = 1.13\,A} \qquad \underline{I_3 = 1.45\,A}$$

With all unknowns filled in, our circuit diagram reads:



Flow of current marked with arrows. You can check these numbers by plugging them back into equations (i) through (vi)

Problem 7-3



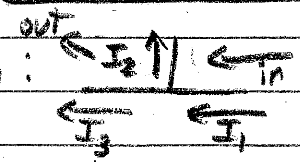
Calculate I_2, I_3, R, and ε... Oh boy. Here we go again.

Kirchhoff's loop rule around the right-hand loop:

$$48\,V - (3A)(12\,\Omega) - (I_2)(6\,\Omega) = 0$$
$$48V - 36V = (I_2)(6\,\Omega)$$
$$12V = (I_2)(6\,\Omega)$$
$$\longrightarrow \underline{I_2 = 2A}$$

For this to work, we need I_2 to flow clockwise — from the point of view of our loop. This means I_2 points **up**.

Kirchhoff's junction rule at the bottom junction:



in = out

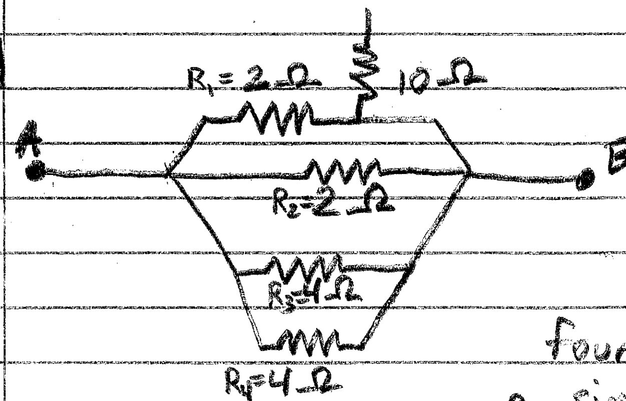
Current in and out balance:

$$I_1 = I_2 + I_3$$
$$3A = 2A + I_3$$
$$\longrightarrow \underline{I_3 = 1A}$$

There. I_2 and I_3.

Homework 7 Solutions

Problem 7-4



$V_{AB} = 6\text{ V}$

Since the $10\,\Omega$ resistor doesn't connect to anything, we can ignore it. The other four resistors are wired in a simple parallel hookup.

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots$$

$$1/R_{total} = 1/(2\Omega) + 1/(2\Omega) + 1/(4\Omega) + 1/(4\Omega)$$

$$1/R_{total} = (0.5\,1/\Omega + 0.5\,1/\Omega + 0.25\,1/\Omega + 0.25\,1/\Omega)$$

$$1/R_{total} = 1.5\,1/\Omega$$

$$\underline{R_{total} = \tfrac{2}{3}\,\Omega = 0.667\,\Omega}$$

Now, current. Six volts acros each resistor.

$V = IR$

$I = V/R$

$I_1 = V/R_1$ $I_2 = V/R_2$ $I_3 = V/R_3$ $I_4 = V/R_4$

$I_1 = (6V)/(2\Omega)$ $I_2 = (6V)/(2\Omega)$ $I_3 = (6V)/(4\Omega)$ $I_4 = (6V)/(4\Omega)$

$\underline{I_1 = 3A}$ $\underline{I_2 = 3A}$ $\underline{I_3 = 1.5A}$ $\underline{I_4 = 1.5A}$

To check: $I_{total} = (3+3+1.5+1.5)A = 9A$

$I_{total} R_{total} = (9A)(2/3\,\Omega) = 6V = V_{AB}$

Problem 7-5

R has units of ohms, or volts per ampere.

C has units of farads, or coulombs per volt.

units of $RC \sim (\cancel{V}/A)(C/\cancel{V})$. Volts cancel

$RC \sim (C/A)$ But one ampere is one coulomb per second...

$RC \sim \left(\frac{C}{C/s}\right)$ Coulombs cancel.

$RC \sim \left(\frac{1}{1/s}\right)$

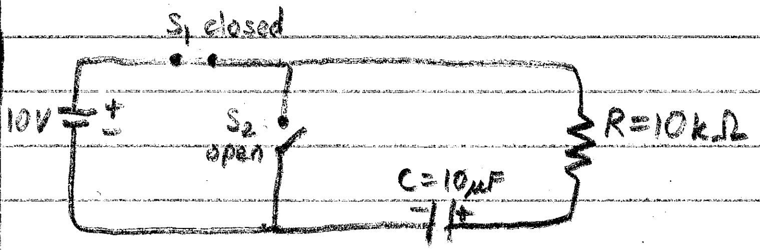
$RC \sim s$ Units of RC are seconds.

$$\left[\; R = \frac{V}{I} \rightarrow \frac{VT}{Q} \qquad C = \frac{Q}{V} \qquad RC \rightarrow \frac{VT}{Q}\cdot\frac{Q}{V} \rightarrow T \;\right]$$

HW 7 Solutions

Problem 7-6 R controls the rate at which the capacitor charges: high R means the capacitor plates receive or release less current, and gain or lose less energe per second. C controls how much charge needs to be received or released; increasing C means it will take more seconds' worth of current flow to fully charge or discharge.
Since the characteristic time of the circuit is linked to how long it takes to charge/discharge the capacitor, both R and C will affect the characteristic time.

Problem 7-7



$$RC = (10\,k\Omega)(10\,\mu F)$$
$$= (10^4\,\Omega)(10^{-5}\,F)$$
$$= 0.1\,s$$

$V_c = V_0(1-e^{-t/RC})$ for a charging capacitor (V_0 is same as ε)

$V = IR$

$I = V/R$

i) At $t = 0.1s$, $V_c = (10V)(1-e^{-(0.1s)/(0.1s)})$ $\quad I_R = (\frac{10V}{R})(e^{-t/RC})$

$V_c = (10V)(1-e^{-1})$ $\quad I_R = (0.001A)(e^{-1})$

$\boxed{V_c = 6.32\,V}$ $\quad \boxed{I_R = 368\,\mu A}$

ii) At $t = 0.3s$, $V_c = (10V)e^{-(0.3s)/(0.1s)}$ $\quad I_R = (\frac{10V}{10k\Omega})(e^{-(0.3s)/(0.1s)})$

$V_c = (10V)(1-e^{-3})$ $\quad I_R = (1mA)(e^{-3})$

$\boxed{V_c = 9.50\,V}$ $\quad \boxed{I_R = 49.8\,\mu A}$

iii) At $t = 1s$, $V_c = (10V)(1-e^{-(1s)/(0.1s)})$ $\quad I_R = (\frac{10V}{10k\Omega})(e^{-(1s)/(0.1s)})$

$V_c = (10V)(1-e^{-10})$ $\quad I_R = (1mA)(e^{-10})$

$\boxed{V_c = 10\,V}$ $\quad \boxed{I_R = 4.54\times10^{-8}\,A = 45.4\,nA}$

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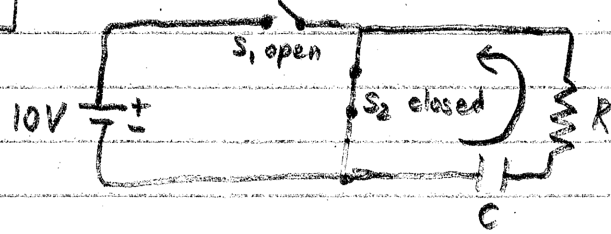
Homework 7 Solutions

Problem 7-8



The capacitor will now discharge via S_2, without being charged via the battery and S_1.

Current flows from 'positive' side of capacitor to 'negative' side: counterclockwise.

$V_c = V_0 e^{-t/RC}$ $\qquad$ $I_R = \frac{V_0}{R}\left(e^{-t/RC}\right)$

V_0 is 10V, that's what the capacitor was held at before we hit the switches.

i) $t = 0.1s$ $\quad V_c = (10V)e^{-(0.1s)/(0.1s)}$ $\qquad I_R = (1mA)\left(e^{-(0.1s)/(0.1s)}\right)$

$\boxed{V_c = 3.68V}$ $\qquad \boxed{I_R = 368\,\mu A}$

ii) $t = 0.3s$ $\quad V_c = (10V)e^{-(0.3s)/(0.1s)}$ $\qquad I_R = (1mA)\left(e^{-(0.3s)/(0.1s)}\right)$

$\boxed{V_c = 0.498V}$ $\qquad \boxed{I_R = 49.8\,\mu A}$

iii) $t = 1s$ $\quad V_c = (10V)e^{-(1s)/(0.1s)}$ $\qquad I_R = (1mA)\left(e^{-(1.0s)/(0.1s)}\right)$

$\boxed{V_c = 454\,\mu V}$ $\qquad \boxed{I_R = 45.4\,nA}$

Problem 7-9 Since ε_1 and ε_2 are the same, what matters is the RC time constants. For the left hand circuit,

$R_1 C_1 = (100\,k\Omega)(10\,\mu F) = 1s$

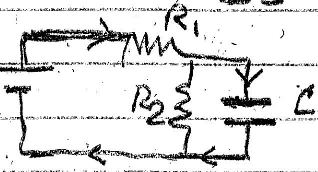
For the right hand circuit,

$R_2 C_2 = (200\,k\Omega)(5\,\mu F) = 1s$

The characteristic times are the same too. Neither capacitor will reach 6V first; they will both reach 6V at the same time.

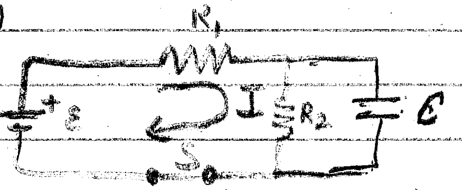
Homework 7 Solutions

Problem 7-10 i) AT $t = 0^+$, $V_C = 0$ SO $V_{R2} = 0$. CURRENT IS CONTROLLED BY R_1, $I = \frac{\varepsilon}{R_1}$
CURRENT FLOWS INTO PLATES WHERE CHARGE ACCUMULATES



NO CURRENT BETWEEN CAPACITOR PLATES AT ANY TIME

ii) A long time later capacitor is fully charged $V_C = V_{R_2}$ and current flows as shown



$$I = \frac{\varepsilon}{R_1 + R_2}$$

$$V_C = V_{R2} = \left(\frac{\varepsilon}{R_1 + R_2}\right) R_2$$

Problem 7-11 Hmm. While charging, $V_c = V_0(1 - e^{-t/RC})$. The question is, how long do we need before V_c is 99% of V_0 or more?

$$V_c = 0.99 V_0 = \cancel{V_0}(1 - e^{-t/RC})$$

$$0.99 = 1 - e^{-t/RC}$$

$$-0.01 = -e^{-t/RC}$$

$$0.01 = e^{-t/RC}$$

Take natural log of both: $-4.61 = -t/RC$

$$4.61(RC) = t$$

So after 4.61 RC time constants pass, V_c will have reached 99% of its equilibrium value.

Homework 7 Solutions

Problem 7-12 How much energy is released as the capacitor discharges?

$$U_E = Q^2/2C \quad \text{(from 'devices based on coulomb E-field' text}$$

We also have $Q = CV$, so

$$U_E = \frac{C^2V^2}{2C} = \frac{1}{2}CV^2$$

$$U_E = \frac{1}{2}(1000\,\mu F)(100\,V)^2$$

$$U_E = \frac{1}{2}(10\ F\cdot V^2)$$

Electrical Equivalent of HEAT

$$\underline{U_E = 5\,J}$$

If we release those five joules into the water, by how much will we heat the water? 20 grams of water...

$$\Delta U = mC\Delta T$$

$$5J = (20g)\left(\frac{1\text{ calorie}}{\text{gram}\cdot K}\right)\left(\frac{4.184\,J}{\text{calorie}}\right)\Delta T$$

Don't forget the joule $\leftrightarrow$ calorie conversion; it's an occupational hazard of using calories for specific heat capacity.

$$5J = (83.68\ J/K)\Delta T$$

$$\Delta T = 0.0598\ K$$

So that's the increase in the temperature of the water.

Problem 7-13 We have I, we want ΔV, we need R... for 1.5 inches of copper cable. Looking it up, for copper, $\rho = 1.68\times10^{-8}\,\Omega\cdot m$ at about room temperature. To calculate the resistance of that chunk of wire...

$$R = \rho(l/A)$$

$$R = (1.68\times10^{-8}\,\Omega\cdot m)\frac{(1.5\text{ inches})(0.0254\text{ m/inch})}{A}$$

$A = \pi r^2$, $r = \left(\frac{1\text{ inch}}{2}\right) = (0.5\text{ in})$

$$R = (6.40\times10^{-10}\,\Omega\cdot m^2)/A$$

$$R = (6.40\times10^{-10}\,\Omega\cdot m^2)/\left\{\pi\left[(0.5\text{ in})\left(0.0254\tfrac{m}{\text{inch}}\right)\right]^2\right\}$$

$$R = (6.40\times10^{-10}\,\Omega\cdot m^2)/(5.07\times10^{-4}\ m^2)$$

$$R = 1.26\times10^{-6}\,\Omega$$

For the segment of cable, this is the resistance.

So $\Delta V = IR = (40\,A)(1.26\times10^{-6}\,\Omega)$

$$\Delta V = 5.02\times10^{-5}\,V$$

$$\boxed{\Delta V = 50.2\,\mu V}$$

This is the voltage across the bird.

Homework 7 Solutions

Problem 7-14 A $\vec{B}$ field will only exert a force on a charge if that charge is moving. An $\vec{E}$ field will exert a force on a charge even if it does not move at all.

If a stationary chare experiences a Force it must be located in an $\vec{E}$-field
If a moving charge experiences a force which is perpendicular to its velocity at all times it must be located in a $\vec{B}$ field.

Problem 7-15 The kinetic energy of the particle, $KE = \frac{1}{2}m(|\vec{v}|)^2$, does not change in circular motion.

We observe that at every point, the magnetic force $\vec{F}_B = q[\vec{v} \times \vec{B}]$ is perpendicular to the direction the particle is moving at that point. A force which is perpendicular to the motion does no work- it does not speed the particle up or slow it down. Therefore, $\vec{B}$ does **no** work as the particle goes half way round the circle, or a quarter way, or all the way, or any other distance around the circle.