1. Raychaudhuri equation & cosmology

The Raychaudhuri equation for a timelike geodesic congruence parameterized by proper time \( t \) is

\[
\frac{d\theta}{dt} = -\frac{1}{3} \theta^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - R_{ab} u^a u^b,
\]

(1)

where \( \theta, \sigma_{ab}, \) and \( \omega_{ab} \) are the expansion, shear, and twist, respectively, and \( u^a \) is the 4-velocity. Consider a homogeneous isotropic universe (i.e. a Robertson-Walker spacetime, possibly with spatial curvature) with scale factor \( a(t) \), comoving energy density \( \rho \), and pressure \( p \).

(a) Use the Raychaudhuri equation for the cosmological geodesic flow (\( u^a \) orthogonal to the homogeneous isotropic surfaces) to derive the second order Friedmann equation,

\[
\ddot{a}/a = -(4\pi/3)(\rho + 3p).
\]

Use the Einstein equation only to evaluate \( R_{ab} u^a u^b \) in terms of \( \rho \) and \( p \).

[Note that a positive pressure adds to the attraction of energy density, while a negative pressure subtracts. If \( p < -\rho/3 \) the universe is accelerating. Vacuum energy is locally Lorentz invariant, hence has \( T_{ab} = \rho_{\text{vac}} g_{ab} \), so \( p = -\rho_{\text{vac}} \). It can therefore produce acceleration.]

(b) Use the Raychaudhuri equation to show that, if \( \rho + 3p > 0 \), a ball of test particles initially at rest with respect to each other, and whose center is at rest with respect to the cosmological fluid, will contract rather than expand. Explain how this is consistent with the cosmological expansion.

2. Hypersurface orthogonal vector fields

A vector field \( V^a \) that is orthogonal to a foliation by hypersurfaces is called hypersurface orthogonal. The hypersurfaces can be characterized as the level sets of some function \( S \). Since \( V_a \) and \( \nabla_a S \) both vanish when contracted with any vector tangent to the constant \( S \) hypersurfaces, they must be proportional, i.e. \( V_a = f \nabla_a S \) for some function \( f \).

(a) Show that if \( V^a \) has this form then

\[
\nabla_{[a} V_{b]} = V_{[a} W_{b]}
\]

(2)

for some co-vector \( W_b \). (The converse, which I do not ask you to prove, is Frobenius’ theorem.)

(b) Show that the condition (2) holds if and only if

\[
V_{[a} \nabla_b V_{c]} = 0.
\]

(3)

(Hint for the if part: Expand out (3) and contract on the index \( c \) with any vector \( X^c \) such that \( X^c V_c \neq 0 \).)
(c) Show that in Minkowski spacetime the Killing fields $\partial_t$ and $\partial_\phi$ are hypersurface orthogonal, but $\partial_t + \Omega \partial_\phi$ (where $\Omega$ is constant) is not.

(d) Show that in the Kerr spacetime, neither of the Killing vector fields $\partial_t$ nor $\partial_\phi$ (in B-L coordinates) are hypersurface orthogonal.

[A stationary spacetime is one with a timelike Killing vector. If the timelike Killing vector is also "hypersurface orthogonal", the spacetime is called static. The Kerr spacetime is stationary, but not static.]

(e) Since $\chi = \partial_t + \Omega H \partial_\phi$ is normal to the Kerr horizon, it is hypersurface orthogonal on the horizon (though not elsewhere). Use this property to show that the square of the surface gravity is given by

$$\kappa^2 = -\frac{1}{2} (\nabla_a \chi_b)(\nabla^a \chi^b).$$

For this purpose use the relation $\chi^a \nabla_a \chi_b = \kappa \chi_b$, which holds on the horizon, to define the surface gravity $\kappa$.

(f) Show that if $u^a$ is hypersurface orthogonal, and is a timelike or spacelike, affinely parametrized geodesic vector field, then the twist vanishes, i.e. $\omega_{ab} := \nabla_{[a} u_{b]} = 0$.

(g) Show that if $k^a$ is hypersurface orthogonal, and is a null, affinely parametrized geodesic vector field, then the twist has the form $\omega_{ab} := \nabla_{[a} u_{b]} = k_{[a} v_{b]}$, where $v^a$ is orthogonal to $k^a$. Conclude that $\omega_{ab} \omega^{ab} = 0$. 