Relativistic Beaming

Consider a source of radiation that emits isotropically in its own rest frame $S_*$. If the source is moving with velocity $v$ in the $x$-direction of an inertial frame $S$, the flux will not be isotropic in $S$ but will rather be concentrated towards the forward direction. This is called relativistic beaming and is very important in high energy astrophysics.

1. (a) A photon with frequency $\omega_*$ travels with angle $\theta_*$ from the $x$-direction in the frame $S_*$. Find the frequency $\omega$ and angle $\theta$ of travel from the $x$-axis in the frame $S$. Show that the angle is given by

$$ \cos \theta = \frac{k_x}{|k|} = \frac{\cos \theta_* + v}{1 + v \cos \theta_*} $$

or (which is simpler for taking the small angle limit)

$$ \tan \theta = \frac{k_y}{k_x} = \frac{\sin \theta_*}{\gamma (\cos \theta_* + v)} $$

where $k$ is the photon wavevector, and we use units with $c = 1$. (Note that one can find the inverse relations by interchanging $\theta$ and $\theta_*$ and replacing $v$ by $-v$.)

(b) To what angle $\theta$ does $\theta_* = \pi/2$ correspond? What angle $\theta_*$ corresponds to $\theta = \pi/2$?

2. Suppose two photons are emitted at angles $\theta$ and $\theta + \delta \theta$ from the moving source, with a time separation $\Delta t_e$, and suppose both photons reach a distant observer at rest in the frame $S$. (Since the observer is distant the angle difference $\delta \theta$ can be neglected.) Show that the time separation of observation of the two photons is given by

$$ \Delta t_o = (1 - v \cos \theta) \Delta t_e $$

where both times are measured in the frame $S$.

3. (a) The specific intensity $I_\omega$ at frequency $\omega$ is defined by $I_\omega = dE/d\omega dt d\Omega$, where $dE$ is the energy in the frequency range $d\omega$ passing in a time $dt$ through a surface subtending a solid angle $d\Omega$. Show that the ratio of specific intensities seen in the two frames is

$$ \frac{I_\omega}{I_{\omega_*}} = (\omega/\omega_*)^3 \left(\gamma (1 - v \cos \theta)\right)^{-3} $$

where $\gamma$ is the usual relativistic gamma factor $(1 - v^2)^{-1/2}$. [Hint: Compare the radiation energy that emerges between the angles $\theta_*$ and $\theta_* + d\theta_*$ during a time $dt_*$ in the frame $S_*$ with the corresponding energy received by the observer in the frame $S$.]

(b) Show that the forward intensity ratio is given by

$$ \frac{I_\omega(0)}{I_{\omega_*}} = \gamma^3 (1 + v)^3 $$

In the limit where $v$ is very close to the speed of light, this memorably becomes $8\gamma^3$. Note that for $\gamma = 10$ this is already of order $10^4$! Sources beamed toward the viewer can appear much brighter than in their rest frame.