Addition of angular momenta:

Rotations in space are implemented on QM systems by unitary transformations
\[ U(R) = \exp(-i \theta J/\hbar), \]
where \( J^i \) are the hermitian generators of rotation.

\( J^i \) are also the angular momentum operators, and are conserved
if the Hamiltonian is invariant under rotations.

Rotation group structure implies \([J^i,J^j] = i\hbar \epsilon^{ijk} J^k\).

Representations: can simultaneously diagonalize \( J_z \) and \( J^2 \), since \([J_z,J^2]=0\).
We analyzed this last semester.
Call the eigenstates \(|jm>\),
where \( J_z |jm> = m |jm> \), \( J^2 |jm> = j(j+1) |jm> \), with \( \hbar=1 \) from now on.
The possible values of \( j \) are
0, 1/2, 1, 3/2, 2, ... and the possible values of \( m \), for a given \( j \), are \( j, j-1, j-2, ..., -j \).
The representation with a given \( j \) is called the "spin-\( j \)" representation,
and it is \( 2j+1 \) dimensional.
These representations are irreducible, in the sense that there is no subspace
that is invariant (mapped into itself) under all rotations.
We can see this from the fact that
\[ J_+ |jm> = \sqrt[j(j+1) - m(m+1)] |jm+1> \quad \text{and} \quad J_- |jm> = \sqrt[j(j+1) - m(m-1)] |jm-1>, \]
where \( J_+ = J_x + i J_y \), \( J_- = J_x - i J_y \),
from which it is clear that by acting with rotations we move through
all the states.

Example: 3d vectors \( V^i \) form the spin-1 rep. The tensor product of two of these
is the rank two tensors like \( V^i W^j \), or more generally, \( T^ij \). These are not irreducible.
Rather the antisymmetric part is by itself irreducible, and three dimensional,
hence another spin-1 rep. The symmetric part is reducible into the part proportional to
the Kronecker delta (trace) and the rest (symmetric trace-free part).
The trace part is the \( j=0 \) rep, the symm tracefree part is \( j=2 \)
(since then \( 2j+1=5=\text{number of independent components of a symmetric tracefree tensor} \)).

Example: \( 1/2 \times 1/2 = 1 + 0 \), example: \( 1 \times 1 = 2 + 1 + 0 \) (this is equivalent to the example above).

Note three different examples of spin-1 rep:
vector, antisymmetric tensor, \( |2p, m=-1,0,1> \) states of H-atom.
I.e., the rep is the abstract structure. Many things can realize it.

General scheme: \( j1 \times j2 \) spanned by basis \( \{|j1m1> |j2m2>\} \). Decomposes into irreducibles.
Find by starting with top \( J_z \) state and working down with lowering operator \( J_- \).
When fill out a rep, go back and start with the next highest top \( J_z \) state, which is the other
linear combination of the two second to two top \( J_z \) states.
This results in
\[ j1 \times j2 = (j_1-j_2) + (j_1+j_2 - 1) + ... + |j_1 - j_2|. \]

The largest spin rep, \( j1+j2 \), starts with top state equal to the product of the two top states \( |j1j1> |j2j2> \).
To see that the smallest spin rep is \( |j1-j2| \), suppose first that \( j1>\equiv j2 \).
The argument I gave in class, cleaned up a bit here, was that the largest degeneracy
that occurs for fixed total m is \(2j_2+1\), so there must be \(2j_2+1\) different irreps in the decomposition.

Working our way down from the \(j_1+j_2\) rep the last one must therefore be the \(j_1-j_2\) rep.

A (sort of) different argument goes as follows. Each state \(|j_1m_1>\) must occur in every
rep, since acting with \(J_+\) and \(J_-\) will eventually introduce it. In particular, \(|j_1j_1>\) must occur.
The smallest total m the state \(|j_1j_1>|j_2m_2>\) can have is if \(m_2\) is as small as possible, \(m_2=-j_2\).
In this case, the total m is \(j_1-j_2\), hence the smallest rep we have is spin- \((j_1-j_2)\).
If \(j_2>j_1\) then reverse the roles, and the smallest rep is spin- \((j_2-j_1)\).

In general, we have that the smallest is spin- \(|j_1-j_2|\). You can check that the total dimension
\((2j_1+1)(2j_2+1)\) is equal to the sum over integer steps from \(j=|j_1-j_2|\) to \(j_1+j_2\) of \((2j+1)\).

\# |jm> = |m1m2><m1m2|jm>, sum on m1,m2 with m=m1+m2.
Similarly, |m1m2>= |jm><jm|m1m2>, where the sum is over j with m=m1+m2 fixed.
The expansion coefficients are the Clebsch-Gordan coefficients. The construction above
shows that they can always be taken to be real, so <m1m2|jm>=<jm|m1m2>*=<jm|m1m2>.

There is still an overall sign ambiguity of the CG coeffs, that is typically fixed by requiring that
the coefficient of \(|m_1=j_1>|m_2=j-j_1>\) in the the expansion of the top state \(|jj>\) of the spin-j rep. is positive,
i.e. \(<j_1,j-j_1|jj>\) is positive. (There is a typo in Baym in the fourth line after (15-40), where it reads
\(m_1=j\) instead of \(m_1=j_1\).)

\# Baym works out the case of \(j \times 1/2\). There is a typo in eqn (15-44), which should have \(m_2 = +/-1/2\).

\# The CG coeffs can be computed by:
- brute force
- Mathematica: ClebschGordan[{j1,m1},{j2,m2},{j,m}]  (Note: I mis-spelled it "Gordon" in class.)
- tables
- recursion relations
- a projection operator method
- amazingly enough, a CLOSED FORM formula has been found by Wigner for all the CG coeffs,
which was given in a more symmetrical form by Racah. See (106.14) of Landau & Lifshitz.
It is so complicated as to be unusable.