Born approximation validity conditions

Integral form of time-independent Schrödinger equation for an incoming plane wave in a potential $V$:

$$\psi(x) = e^{ik \cdot x} - \frac{m}{2\pi\hbar^2} \int d^3x' \frac{e^{ik|x-x'|}}{|x-x'|} V(x') \psi(x')$$  \hspace{1cm} (1)

The (first) Born approximation consists of replacing $\psi(x')$ in the integrand by $e^{ik \cdot x'}$:

$$\psi_{FBA}(x) = e^{ik \cdot x} - \frac{m}{2\pi\hbar^2} \int d^3x' \frac{e^{ik|x-x'|}}{|x-x'|} V(x') e^{ik \cdot x'}.$$  \hspace{1cm} (2)

A sufficient condition for this to be a good approximation is that the difference between $\psi(x)$ and $e^{ik \cdot x}$ be small in the region $R$ where the integral receives important contributions; i.e. if for all $x \in R$,

$$\frac{m}{2\pi\hbar^2} \left| \int d^3x' \frac{e^{i(k|x-x|+k \cdot x')}}{|x-x'|} V(x') \right| \ll 1.$$  \hspace{1cm} (3)

In terms of $r = x' - x$, and with $x$ chosen as the coordinate origin, this can be written more simply as

$$\frac{m}{2\pi\hbar^2} \left| \int d^3r \frac{e^{ikr}}{r} \frac{1}{r} V(r) \right| \ll 1.$$  \hspace{1cm} (4)

Let’s now suppose the potential is negligible outside some radius $a$, and consider separately the low energy case $ka \ll 1$ and the high energy case $ka \gg 1$.

In the low energy case $ka \ll 1$, the exponential doesn’t oscillate enough to significantly suppress the integral, so the condition (4) becomes independent of $k$,

$$\frac{m}{2\pi\hbar^2} \left| \int d^3r \frac{1}{r} V(r) \right| \ll 1.$$  \hspace{1cm} (5)

If the potential is characterized by a largest absolute value $V_{\text{max}}$, a sufficient condition is thus

$$V_{\text{max}} \ll \frac{\hbar^2}{ma^2} \quad \text{weakness condition}$$  \hspace{1cm} (6)

This can be thought of as saying that the potential energy is much smaller than the kinetic energy associated with the particle being localized in the potential. A similar condition can be derived even if $V(r)$ has no maximum, as long as the integrand has a maximum. For instance, although $V(r) = \frac{1}{r} e^{-r/a}$ has no maximum, $rV(r)$ does have one. Note that the weakness condition (6) is sufficient for any $ka$, since oscillations of the exponential in (4) could only make the integral smaller.

In the high energy case $ka \gg 1$, oscillations make the integral smaller, so the Born approximation can be valid even if the potential is not “weak” in the sense of (6). The exponential is unity for $r$ opposite to $k$, and is not oscillating rapidly only in the solid angle where $ka(1 + \cos \theta) \lesssim 1$, i.e. where $\delta \theta^2 \lesssim 1/ka$. This means the integral is of order $1/ka$ times what it is in the long wavelength case $ka \ll 1$. Thus a sufficient condition for validity is

$$\frac{V_{\text{max}}}{\hbar^2/ma^2} \ll ka \quad \text{high energy condition}$$  \hspace{1cm} (7)

This is equivalent to the condition $\int V \, dt \ll \hbar$ we derived when treating the scattering by first order time-dependent perturbation theory.